Gluing arguments.

Gluing

By *gluing* we mean the process of pasting a collection of data that are identified on open subsets. In what follows I means an indexing set and i, j, k denotes generic elements of this set.
The following is a purely topological argument.

Lemma. *Let (U_i) be a collection of topological spaces with U_{ij} being an open set of U_i for each i, with $U_{ii} = U_i$. Suppose furthermore that we have isomorphisms $\varphi_{ji} \colon U_{ij} \to U_{ji}$ satisfying the cocyle condition: for all i, j, k we have*

$$\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}.$$

Then there exists a topological space X with open embeddings $\psi_i \colon U_i \to X$ such that

(1) $\bigcup_i \psi_i(U_i) = X$
(2) $\psi_i(U_{ij}) = \psi_j(U_{ji}) = \psi_i(U_i) \cap \psi(U_j)$
(3) $\psi_i \varphi_{ij} = \psi_j$.

This topological space is unique up to unique isomorphism because it satisfies the following universal property. A map from $f \colon X \to Y$ is the same as a collection of maps from $f_i \colon U_i \to Y$ such that $f_i \varphi_{ij} = f_j$.

Proof. We take the the quotient of

$$\bigsqcup U_i$$

using the equivalence relation given by $x_i \in U_i$ is the same as $x_j \in U_j$ if and only if $x_i \in U_{ij}$ and $\varphi_{ji}(x_i) = x_j$. This is an equivalence relation due to the cocycle condition. Denote by $\psi_i \colon U_i \to X$ the natural maps. Conditions (1)-(2)-(3) now hold by construction, and ψ_i is injective.
We endow the quotient with the following topology: a set V is open if and only if $\psi_i^{-1}(V)$ is open in U_i for all i. Therefore we see that ψ_i is an open embedding, indeed $\psi_j^{-1} \psi_i(U_i) = \varphi_{ji}(U_{ij})$ is open in U_j. □

Now we can extend the gluing argument to locally ringed spaces.

Lemma. *Let $(U_i, \mathcal{O}_{U_i})$ be a collection of locally ringed spaces with U_{ij} being an open set of U_i for each i, with $U_{ii} = U_i$. Suppose furthermore that we have isomorphisms of locally ringed spaces $\varphi_{ji} \colon (U_{ij}, \mathcal{O}_{U_i|U_{ij}}) \to (U_{ji}, \mathcal{O}_{U_j|U_{ji}})$ satisfying the cocyle condition: for all i, j, k we have*

$$\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}.$$

Then there exists a ringed space $(X, \mathcal{O}_X)$ with open embeddings $\psi_i \colon U_i \to X$ such that

(1) $\bigcup_i \psi_i(U_i) = X$
(2) $\psi_i(U_{ij}) = \psi_j(U_{ji}) = \psi_i(U_i) \cap \psi(U_j)$
(3) $\psi_i \varphi_{ij} = \psi_j$.

This locally ringed space is unique up to unique isomorphism because it satisfies the following universal property. A map from $f\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ of locally ringed spaces is the same as a collection of maps from $f_i\colon (U_i,\mathcal{O}_{U_i}) \to (Y,\mathcal{O}_Y)$ such that $f_i\varphi_{ij} = f_j$.

Proof. Let X be the topological space constructed in the last lemma. Namely a map $f\colon X \to Y$ is the same as a collection of maps $f_i\colon U_i \to Y$ with $f_i\varphi_{ij} = f_j$. The bijection is given by $f \mapsto f \circ \psi_i$. Note that in the data we have isomorphisms in $\mathrm{Sh}(U_{ij})$

$$\varphi_{ij}^\sharp\colon \mathcal{O}_{U_i|U_{ij}} \to \varphi_{ij*}\mathcal{O}_{U_j|U_{ji}}.$$

If we apply ψ_{i*} we get isomorphisms in $\mathrm{Sh}(\psi_i(U_{ij})) = \mathrm{Sh}(\psi_j(U_{ji}))$

$$\psi_{i*}(\varphi_{ij}^\sharp)\colon \psi_{i*}\mathcal{O}_{U_i|U_{ij}} \to \psi_{i*}\varphi_{ij*}\mathcal{O}_{U_j|U_{ji}} = \psi_{j*}\mathcal{O}_{U_j|U_{ji}}.$$

Denote this isomorphisms by σ_{ij}. These isomorphisms will satisfy the cocyle condition.
Now $(\psi_{i*}\mathcal{O}_{U_i})$ together with (σ_{ij}) is a collection of sheaves on the opens $\psi_i(U_i)$ of X and isomorphisms as in the *gluing sheaves exercise*. Therefore, let $\mathcal{O}_X$ be the sheaf of rings on X which is the gluing of the preceding data.
As for the universal property, on the topological side it follows from the last lemma. On the sheaves side, a map

$$f^\flat\colon f^*\mathcal{O}_Y \to \mathcal{O}_X$$

is the same by universal property of $\mathcal{O}_X$ as a gluing (see the exercise about gluing sheaves) as a collection of maps

$$f_i^\flat\colon f^*\mathcal{O}_Y \to \psi_{i*}\mathcal{O}_{U_i}$$

with $\sigma_{ij}f_i^\flat = f_j^\flat$. But by adjunction this is the same as a collection of maps

$$f_i^*\mathcal{O}_Y = (f \circ \psi_i)^*\mathcal{O}_Y \to \mathcal{O}_{U_i}$$

compatible with φ_{ij}'s which is what is in the sheaf data of a ringed spaces map $f_i\colon (U_i,\mathcal{O}_{U_i}) \to (Y,\mathcal{O}_Y)$ such that $f_i\varphi_{ij} = f_j$. □

Remark. If each $(U_i,\mathcal{O}_{U_i})$ is a scheme, then $(X,\mathcal{O}_X)$ is a scheme.

Remark. Any scheme is a gluing of affine schemes. Namely, by hypothesis a scheme $(X,\mathcal{O}_X)$ is a locally ringed space such that there are open subsets U_i with isomorphisms

$$\varphi_i\colon (U_i,\mathcal{O}_{U_i}) \to \mathrm{Spec}(A_i).$$

Therefore $(X,\mathcal{O}_X)$ is the gluing of $\mathrm{Spec}(A_i)$ with cocycles $\varphi_{ij} = \varphi_i\varphi_j^{-1}$. In particular to define a map $f\colon X \to Y$ to another scheme Y is the same as defining a collection of maps from $f_i\colon \mathrm{Spec}(A_i) \to Y$ which $f_i\varphi_{ij} = f_j$.

Covering by affine schemes

In a scheme $(X,\mathcal{O}_X)$ the intersection of two affine opens need not to be affine. However the following lemma holds.

Lemma. *Let U and V two open affines of a scheme X. Say that $\phi\colon \mathrm{Spec}(A) \to U$ and $\psi\colon \mathrm{Spec}(B) \to V$ are isomorphisms. Then there exists a covering of $U \cap V$ by open affines such that the intersection of each of this open affines* *is affine. More precisely, there exists elements f_i and g_i in A and B respectively with*

$$U \cap V = \bigcup \phi(D(f_i)) = \bigcup \psi(D(g_i)).$$

Proof. It suffices to show that for every point $x \in U \cap V$ there is $f \in A$ and $g \in B$ with

$$\phi(D(f)) = \psi(D(g)).$$

Let $x \in \phi(D(f)) \subset U \cap V$. Let $g \in B$ with

$$\psi^{-1}(x) \in D(g) \subset \psi^{-1}(\phi(D(f))) \subset \operatorname{Spec}(B).$$

Therefore $\phi^{-1}\psi$ induces a map of rings $B \to A_f$ which send g to an element $g'/f^n \in A_f$. It follows that $\phi(D(fg')) = \psi(D(g))$.

□

This can be useful in some gluing arguments.

Example. We want to show that $\operatorname{Spec}(A_{red}) \to \operatorname{Spec}(A)$ is the reduction in the category of schemes using a "gluing type argument". We proceed in three steps.

(1) For $X \to \operatorname{Spec}(A)$ with X affine, it follows by duality between affine schemes and rings that it holds for affine schemes.
(2) Suppose that X is a scheme which admits an affine open cover (U_i) with intersections being affine. From a map $X \to \operatorname{Spec}(A)$ we get a collection of maps $U_i \to \operatorname{Spec}(A)$ by restriction. By the preceding point we have a unique morphism $f_i : U_i \to \operatorname{Spec}(A_{red})$ with the desired property. The intersection of U_i and U_j being affine by hypothesis, the two maps f_i and f_j necessarily agree on $U_i \cap U_j$ by unicity in the universal property. Therefore there is a unique map $f : X \to \operatorname{Spec}(A_{red})$ satisfying the requirement.
(3) Now for a general scheme X, we can cover it by affine schemes. They will intersect in a scheme which satisfies the requirement of (2). Therefore we get a collection of map $f_i : U_i \to \operatorname{Spec}(A_{red})$ which will agree on $U_i \cap U_j$ by the universal property showed in (2).