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AG II - Schemes and sheaves

Exercises – week 9

Exercise 1. *Dual.* Let $\mathcal{E}$ be locally free sheaf of finite rank[1] on a ringed space $(X, \mathcal{O}_X)$ and $\mathcal{F}$ an $\mathcal{O}_X$-module. We define $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$. Show that there is a natural isomorphism $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$.

Exercise 2. *Compatibilities between f^*, f_* and $\otimes$.* Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(1) Let $\mathcal{G}$ and $\mathcal{H}$ be sheaves of $\mathcal{O}_Y$-modules. Show that there is a natural isomorphism
$$f^*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{H}) \cong f^*(\mathcal{G}) \otimes_{\mathcal{O}_X} f^*(\mathcal{H}).$$
(2) (*Projection formula.*) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{E}$ be a finite locally free sheaf on $\mathcal{O}_Y$. Show that there is a natural isomorphism
$$\mathcal{E} \otimes_{\mathcal{O}_Y} f_*\mathcal{F} \to f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

Exercise 3. *Points of projective spaces, naturally.* Let k be a field and V be a finite dimensional vector space. Define
$$\mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}(V^\vee)).$$
Let $l \subset V$ be a k-linear subspace of dimension 1.

(1) Show that if V has dimension 1, then the natural map
$$\mathbb{P}(V) \to \mathrm{Spec}(k)$$
is an isomorphism. Therefore by functoriality of Proj when V is a general finite dimensional vector space, deduce that the map induced by $l \subset V$[2]
$$\mathbb{P}(l) \to \mathbb{P}(V)$$
defines k-rational point.
(2) Moreover, show this defines a bijection
$$\mathbb{P}(V)(k) \leftrightarrow \{l \subset V \mid l \text{ is a one dimensional linear subspace of } V\}.$$

Exercise 4. *Tautological line bundle.*
Let A be a ring and M be a finite projective module. Consider[3]
$$\mathbb{P}(M) := \mathrm{Proj}(\mathrm{Sym}(M^\vee)) \xrightarrow{\pi} \mathrm{Spec}(A).$$

[1] For $n \in \mathbb{N}$ a locally free sheaf of rank n is an $\mathcal{O}_X$-module which is locally isomorphic to $\mathcal{O}_U^{\oplus n}$ where U ranges in an open cover of X.
[2] So Proj of the induced map $\mathrm{Sym}(V^\vee) \to \mathrm{Sym}(l^\vee)$.
[3] Called the projective bundle associated to M.

Let $c \in M \otimes M^\vee$ be the canonical element corresponding to the identity along the natural isomorphism $M \otimes M^\vee \cong \mathrm{Hom}_A(M, M)$.

(1) Show that the sub-module generated on $D_+(\varphi)$ by

$$c/\varphi \in \pi^* M(D_+(\varphi)) = \mathrm{Sym}(M^\vee)_{(\varphi)} \otimes M$$

defines a *line bundle* that we denote $\mathcal{O}(-1) \subset \pi^* M$. Show that this line bundle is isomorphic to the one defined in the lecture.

(2) Show that $\mathbb{V}(\mathcal{O}(-1))$ is a closed subscheme of $\mathbb{V}(\pi^* M) = \mathbb{V}(M) \times_A \mathbb{P}(M)$.

(3) *Tautological line bundle.* For any point $x \in \mathrm{Spec}(A)$ show that when base-changing to $\mathrm{Spec}(k(x))$ we obtain the following closed subset on $k(x)$-rational points

$$\mathbb{V}(\mathcal{O}(-1))(k(x)) = \{(v, l) \subset M(x) \times \mathbb{P}(M(x))(k(x)) \mid v \in l\}.$$

Exercise 5. *Euler sequence.* Let A be a ring and M a finite projetive A-module.

(1) *Directional derivative.* For a $v \in M$, show that there is a unique A-derivation

$$\frac{\partial}{\partial v} \colon \mathrm{Sym}(M^\vee) \to \mathrm{Sym}(M^\vee)$$

which is equal to the evaluation at v on elements of degree 1. If M is free, if (e_i) and (x_i) denotes a basis and a dual basis respectively, and $v = \sum \lambda_i e_i$, show that

$$\frac{\partial}{\partial v} = \sum_i \lambda_i \frac{\partial}{\partial x_i}.$$

(2) For $\varphi \in M^\vee$, show that $\frac{\partial}{\partial v}$ uniquely extends to an A-derivation

$$\frac{\partial}{\partial v} \colon \mathrm{Sym}(M^\vee)_\varphi \to \mathrm{Sym}(M^\vee)_\varphi.$$

Deduce that $\frac{\partial}{\partial v}$ defines an A-derivation[4],

$$\frac{\partial}{\partial v} \colon \mathrm{Sym}(M^\vee)_{(\varphi)} \to \mathrm{Sym}(M^\vee)(-1)_{(\varphi)}.$$

(3) Denote by $\pi \colon \mathbb{P}(M) \to \mathrm{Spec}(A)$ and $\mathcal{T}^1_{\mathbb{P}(M)|A} = \left(\Omega^1_{\mathbb{P}(M)|A}\right)^\vee$. Deduce from the above that there is a $\mathcal{O}_{\mathbb{P}(M)}$-linear map

$$\frac{\partial}{\partial(-)} \colon \pi^* M \to \mathcal{T}^1_{\mathbb{P}(M)|A}(-1).$$

Hint: $\mathcal{T}^1_{\mathbb{P}(M)|A}(-1) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}(M)}}(\Omega^1_{\mathbb{P}(M)|A}, \mathcal{O}(-1))$[5]*. Use the universal property of* $\Omega^1_{\mathbb{P}(M)|A}$ *on affines* $D_+(\varphi)$.

[4] where $(-)(-1)$ denotes tensoring by $\mathcal{O}(-1)$.

[5] Because in general if $\mathcal{F}$ is finite locally free and $\mathcal{G}$ is a sheaf of $\mathcal{O}$-modules, then $\mathcal{F}^\vee \otimes \mathcal{G} \cong \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$

(4) *Euler sequence.* Show that there is an exact sequence of $\mathcal{O}_{\mathbb{P}(M)}$-locally free sheaves

$$0 \to \mathcal{O}(-1) \to \pi^* M \xrightarrow{\frac{\partial}{\partial(-)}} \mathcal{T}^1_{\mathbb{P}(M)|A}(-1) \to 0$$

where the first arrow is the canonical inclusion $\mathcal{O}(-1) \to \pi^* M$ and the second is the arrow above. *Hint: show the sequence is exact when restricted to standard opens.*