

### Exercises – week 8

**Remark.** *The Jacobian criterion for hypersurfaces.* In the following, we explain the so called *Jacobian criterion*, in a simple case. Let  $k$  be a field. Consider  $f \in k[x_1, \dots, x_n]$  a non-zero polynomial. Let

$$R = k[x_1, \dots, x_n]/f.$$

Denote also by  $I = (f)$ . Note that the unique map of  $k[x_1, \dots, x_n]$ -modules  $k[x_1, \dots, x_n] \rightarrow I$  sending 1 to  $f$  is an isomorphism, so that  $I$  is a free  $k[x_1, \dots, x_n]$ -module of rank 1. Therefore we have a map of free  $R$ -modules

$$(1) \quad R \otimes_{k[x_1, \dots, x_n]} I = I/I^2 \rightarrow \bigoplus_{i=1}^n R dx_i$$

sending the generator  $f$  to the exterior derivative

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \in \bigoplus_{i=1}^n R dx_i.$$

Then  $R$  is smooth over  $k$  if and only if the map from Equation (1) is injective with a projective cokernel. Indeed: the cokernel is  $\Omega_{R/k}$ . Linear algebra translates this condition to: for every  $\mathfrak{p} \in \text{Spec}(R)$  the fiber at  $\mathfrak{p}$  of the the map in Equation (1) is not zero. This is equivalent to saying that the closed subscheme  $V(\frac{\partial f}{\partial x_i})_{i=1}^n$  in  $\text{Spec}(R)$  is empty. Indeed points of this closed subscheme are precisely points where  $\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i = 0$ . In the end we can reformulate that  $R$  is smooth if and only if the following closed subscheme of  $\mathbb{A}_k^n$  is empty:

$$V\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, f\right) = \emptyset.$$

**Exercise 1.** *Jacobian criterion.* Let  $k$  be an algebraically closed field. Are the following schemes smooth over  $k$ ?

- (1)  $V_+(XZ - Y^2) \subseteq \mathbb{P}_k^2$
- (2)  $V(xz - y^2) \subseteq \mathbb{A}_k^3$
- (3)  $V_+(XZ - Y^2), V_+(YW - Z^2), V_+(XW - YZ) \subseteq \mathbb{P}_k^3$
- (4)  $V(y^2 - x(x-1)(x+1)) \subseteq \mathbb{A}_k^2$

*Hint: Be careful about the characteristic of  $k$ !*

**Exercise 2.** *Regularity vs smoothness.* Let  $k = \mathbb{F}_p(t)$ , and consider  $X = V(x^p - t) \subseteq \mathbb{A}_k^1$ . Show that  $X$  is regular.<sup>1</sup>

<sup>1</sup>Recall that a Noetherian local ring  $(R, \mathfrak{m})$  is regular if and only if  $\dim(R) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ . Do you see why this first question is silly?

However, let  $k' = \mathbb{F}_p(t^{1/p}) \supseteq k$ . Then show that the base change  $X_{k'}$  is not regular.

Was  $X$  smooth over  $k$  in the first place?

**Exercise 3.** *Separable extensions and differentials.* Let  $k$  be a field and  $l$  a finite extension. Show that  $\Omega_{l|k}^1 = 0$  if and only if  $l$  is a separable extension.

**Exercise 4.** *Relative Spec.* Let  $S$  be a scheme. Let  $\mathcal{A}$  be a quasi-coherent  $\mathcal{O}_S$ -algebra. This means that it is a sheaf  $\mathcal{O}_S$ -algebras which is quasi-coherent as an  $\mathcal{O}_S$ -module.

- (1) Let  $V \subset U \subset S$  two open affines. Show that the diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{A}(V)) & \longrightarrow & \mathrm{Spec}(\mathcal{A}(U)) \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \end{array}$$

is cartesian.

- (2) Let  $X = \bigcup U_i$  be an affine cover. Deduce that we can glue the schemes  $(\mathrm{Spec}(\mathcal{A}(U_i)))$  to an  $S$ -scheme

$$\underline{\mathrm{Spec}}_S(\mathcal{A}) \rightarrow S.$$

- (3) Show that  $\underline{\mathrm{Spec}}_S(\mathcal{A})$  satisfies the following universal property in the category of  $S$ -schemes. If  $f: T \rightarrow S$  is an  $S$ -scheme then a  $S$ -morphism  $T \rightarrow \underline{\mathrm{Spec}}_S(\mathcal{A})$  is the same as a morphism of  $\mathcal{O}_T$ -algebras  $f^*\mathcal{A} \rightarrow \mathcal{O}_T$ . Deduce that  $\underline{\mathrm{Spec}}_S(\mathcal{A})$  is independent of the affine cover for the construction.
- (4) Let  $f: X \rightarrow Y$  be an affine morphism of schemes. Show that there is a natural isomorphism of  $Y$ -schemes  $X \cong \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$ .
- (5) Let  $\mathcal{E}$  be a locally free sheaf of finite rank on  $S$ . We define the *schematic vector bundle associated to  $\mathcal{E}$*  by

$$\mathbb{V}(\mathcal{E}) = \underline{\mathrm{Spec}}_S(\mathrm{Sym}(\mathcal{E}^\vee))$$

where the  $\mathcal{O}_S$ -algebra  $\mathrm{Sym}(\mathcal{E}^\vee)$  denotes the free  $\mathcal{O}_S$ -algebra generated by  $\mathcal{E}^\vee$ .<sup>2</sup> Show that a  $S$ -morphism from  $f: T \rightarrow S$  to  $\mathbb{V}(\mathcal{E})$  is the same as a global section of  $f^*(\mathcal{E})$ , i.e. an element of  $f^*(\mathcal{E})(T)$ .

- (6) Show that there is always a canonical section of  $p: \mathbb{V}(\mathcal{E}) \rightarrow S$  which correspond to  $0 \in \mathcal{E}(S)$  which defines a closed subscheme of  $\mathbb{V}(\mathcal{E})$  isomorphic to  $S$ . We call this closed subscheme the *zero section* of  $\mathbb{V}(\mathcal{E})$ .

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<sup>2</sup>It's a gluing of the usual construction in linear algebra.

**Exercise to hand in.** *Resolving singularities of a surface.* (Due Wednesday November 19, 12:00) Please write your solution in  $\text{\TeX}$ .

Let  $k$  be a field of characteristic different than 2. Consider

$$S = \text{Spec} \left( \frac{k[x, y, z]}{(x^2 + y^2 + z^2)} \right).$$

- (1) Show that  $S$  is not smooth using the Jacobian criterion.
- (2) Consider  $S' \rightarrow S$  the blow-up of  $S$  at  $V(x, y, z)$ . Using the Jacobian criterion on each standard blow-up affine chart<sup>3</sup> show that  $S'$  is smooth over  $k$ .
- (3) On each standard blow-up affine chart  $U \subset S'$ , show that

$$\text{Der}_k(\mathcal{O}_{S'}(U), \mathcal{O}_{S'}(U))$$

is a free  $\mathcal{O}_{S'}(U)$ -module of rank 2, and find two explicit derivations which are generators of this free module.

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<sup>3</sup>Take the symmetry of the problem to your advantage.