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AG II - Schemes and sheaves

# Exercises – week 7

**Exercise 1.** *Regular proper curves are projective.* Let $k$ be a field. In this exercise, we say that a 1-dimensional finite type separated normal scheme over $k$ is a *regular curve over* $k$. Recall that a local Noetherian normal ring is a DVR.[1] We say that a *projective* $k$*-scheme* is a closed sub-scheme of $\mathbb{P}^n_k$ for some $n \geq 1$.

(1) *Extension principle.* Let $C$ be regular curve over $k$. Let $x \in C$ be a closed point and $U = C \setminus \{x\}$. Let $Y \to \operatorname{Spec}(k)$ be proper. Show, using the valuative criterion for properness, that any map $U \to Y$ of $k$-schemes extend uniquely to a map $C \to Y$.

(2) Show that any proper regular curve $C$ is a $k$-projective scheme. You can proceed as follows.

  (a) Let $(U_i)_{i=1}^n$ be an open cover of $C$ by affine (regular) curves. Show that for each $i$ there is a morphism of $k$-schemes $U_i \to Y_i$ which is an open immersion where $Y_i$ is a $k$-projective scheme which is integral, separated and finite type.

  (b) *The product of projective schemes is projective.* Let
$$Y = Y_1 \times_k \cdots \times_k Y_n.$$
Show that $Y$ is a $k$-projective scheme.

  (c) Using the *extension principle* show that the natural map
$$\bigcap_{i=1}^{n} U_i \to Y$$
extends uniquely to a map $\varphi\colon C \to Y$.

  (d) Using Exercise 4, consider $Z := \varphi(C) \subset Y$ the closed subscheme with the reduced structure of $Y$. Show that the induced map $\psi\colon C \to Z$ is an isomorphism and conclude.
*Hint: Using that the map $C \to Z$ is surjective by construction, and that for every $i$*
$$U_i \to Z \to Y_i$$
*is an open immersion, show that for every $c \in C$ we have that the map $\mathcal{O}_{Z,f(c)} \to \mathcal{O}_{C,c}$ is an isomorphism. Conclude that $Z$ is a regular curve and that for any $i$, $U_i \to Z$ is an open immersion. Therefore extend the inclusion $U_i \to C$ map to a map $Z \to C$ and show that this will be an inverse map to the map constructed above.*

[1] See Tag 00PD for example.

**Exercise 2.** *An open of an affine is not necessarily affine.* Let $R$ be a non-zero ring. Show that $U = \mathrm{Spec}(R[x,y]) \setminus V(x,y)$ is not affine.
*Hint: compute $\mathcal{O}(U)$ using an appropriate cover and the sheaf property.*

**Exercise 3.** *Intersection of affine schemes.* Let $X$ be a scheme and $U, V \subset X$ be open affine sub-schemes.

(1) Show that if $X$ is separated then $U \cap V$ is affine.
*Hint: Show that $U \cap V \cong X \times_{X \times X} (U \times V)$.*
(2) Show that $U \cap V$ is not necessarily affine if $X$ is not separated.
*Hint: remember this open of an affine which is not affine? Play with this.*

**Exercise 4.** *A map from a proper scheme to a separated scheme is closed.* Let $f\colon X \to Y$ be a map of $S$-schemes. Suppose that $Y \to S$ is separated.

(1) Show that the graph $(\mathrm{id}, f) = \Gamma_f\colon X \to X \times_S Y$ is a closed immersion.
(2) Let $Z \subset X$ a closed subscheme proper over $S$. Show that $f_{|Z}$ is closed.

**Remark.** This fact is analogue to the topological result that a continuous map from a compact topological space to a Hausdorff space is always closed.

**Exercise 5.** *Morphisms into separated schemes.* Let $S$ be a scheme. Let $X \to S$ and $Y \to S$ be $S$-schemes. Suppose that $X$ is reduced and $Y \to S$ separated. Show that two morphisms of $S$-schemes

$$f_1, f_2\colon X \to Y$$

that coincide an open dense subset of $X$ are equal.
Give counter-examples if one of the hypotheses is dropped.

**Remark.** This fact is analogue to the topological result that if two continuous morphisms to a Hausdorff space agree on a open dense then they actually agree everywhere.