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EPFL, fall semester 2025
AG II - Schemes and sheaves

Exercises – week 6

Exercise 1. *Blow-ups.* Let R be a ring and $I \subset R$ an ideal. We define the blow-up of $\mathrm{Spec}(R)$ at $V(I)$ to be the map ($I^0 = R$)

$$b\colon \mathrm{Bl}_I = \mathrm{Proj}(\bigoplus_{n\geq 0} I^n) \to \mathrm{Spec}(R).$$

The *exceptional divisor* of the blow-up is the closed subscheme of $\mathrm{Proj}(\bigoplus_{n\geq 0} I^n)$

$$E = V_+(\bigoplus_{n\geq 0} I^{n+1}).$$

(1) Show that b defines an isomorphism of schemes

$$b\colon \mathrm{Bl}_I \setminus E \to \mathrm{Spec}(R) \setminus V(I).$$

(2) Let A be a ring, $R = A[x_0, \dots, x_n]$ and $I = (x_0, \dots, x_n)$. Show that $E \cong \mathbb{P}^n_A$.
Remark. Let us introduce a bit of intuition. Points (1) and (2) subsume the key philosophy of blow-ups. First of all a blow-up is a map which is an isomorphism outside of a the fiber closed subscheme $V(I)$. We will see later that in nice cases I/I^2 is to be interpreted as the *conormal bundle* of $V(I)$ in $\mathrm{Spec}(R)$ (*i.e.* tangent vectors going out of $V(I)$). Therefore E can be interpreted as the projective space of the vector space of directions outside of $V(I)$. Meaning that for each direction going outside of $V(I)$, there is a corresponding point in E. For example, in the actual computation above, the exceptional divisor E is the space of lines through the origin in $\mathbb{A}^{n+1}$, i.e. $\mathbb{P}^n$.

(3) *Standard blow-up charts.* Consider the same setting as in the last item. Show that Bl_I can be identified as the scheme

$$V_+(x_iY_j - x_jY_i)$$

inside $\mathbb{A}^{n+1}_A \times \mathbb{P}^n_A = \mathrm{Proj}(A[x_0, \dots, x_n] \otimes_A A[Y_0, \dots, Y_n])$ where the grading is taken to be the Y-grading.

Remark. See here for a representation of the blow-up at (x_0, x_1) of $\mathbb{A}^2$ using the standard charts. The projection to the x, y-plane is a bijection outside of the pre-image of the origin which is a line.

Exercise 2. *Strict transforms.* Let R be a ring. Let I be an ideal and $b\colon \mathrm{Bl}_I \to \mathrm{Spec}(R)$ the blow-up at the ideal I. Let $J \subset R$ be another ideal. We define the *strict transform of* $V(J)$ to be the blow-up of $V((I+J)/J)$ in $\mathrm{Spec}(R/J)$.

(1) Show that St_J can be identified with be the closed subscheme of Bl_I

$$V_+\left(\bigoplus_n I^n \cap J\right).$$

(2) Show that b induces an isomorphism

$$b\colon \mathrm{St}_J \setminus E \to V(J) \setminus V(I).$$

(3) *Resolving a singularity.* Let k be a field. Compute the strict transform with $R = k[x_0, x_1]$, the ideal $I = (x_0, x_1)$ and $J = (x_1^2 - (x_0^3 + x_0^2))$. Use the standard blow-up charts. Show that this strict transform is regular.

Remarks. The equation of the last item is the equation of a *nodal curve* which is a type of singular curve. See here for a representation. The tangent space at the origin has dimension 2, which is why the curve is not regular. Since the blow-up at a point replaces a point by "directions out of the point", it is no surprise that blowing up a node at its nodal point removes the singularity.

Exercise 3. *Examples and computations of blow-ups.* Let k be an algebraically closed field. You can use the following.
Let $A = k[x_1, \dots, x_n]/(f)$. Denote by $\partial_i f$ the derivative of f with respect to x_i. Then

$$\mathrm{Spec}(A) \text{ is regular } \iff V(f, \partial_1 f, \cdots, \partial_n f) = \emptyset.$$

Moreover $V(f, \partial_1 f, \cdots, \partial_n f)$ consists exactly of the non-regular points of $\mathrm{Spec}(A)$.

(0) Let R be a ring. Show that if $I = (f_0, \dots, f_n)$ is generated by a regular sequence then $\mathrm{Bl}_I = V_+(X_i f_j - X_j f_i)$ in $\mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times \mathrm{Spec}(R)$. (Use the lemmas in the blow-ups document from moodle)
(1) Show that blow-up of (x^2, y) in $\mathrm{Spec}(k[x, y])$ is not regular. What are the regular points?[1]
(2) Show that $X = \mathrm{Spec}(k[x, y, z, w]/(xy - zw))$ is not regular. What are the regular points?
(3) Show that blow-ups of X at (x, y, z, w) and (x, z) are regular. We denote these blow-ups by $X_1 \to X$ and $X_2 \to X$.
Remark. This is another example where blow-ups resolves (=removes) singularities.
(4) Compute fibers of (x, y, z, w) of $X_1 \to X$ and $X_2 \to X$.

Exercise 4. *Introduction to derivations.* Let R be a ring and $R \to A$ an R-algebra. Let M be an A-module.
Definition. An *R-linear derivation* is a map of A-modules

$$d\colon A \to M$$

[1] This investigation can be used to show that this blow-up is normal.

such that for all $f, g \in A$

$$d(fg) = fd(g) + gd(f) \quad \text{(Leibniz rule)}$$

We denote this set by $\mathrm{Der}_R(A, M)$.

(1) Show that $d(r) = 0$ for any $r \in R$.

(2) Consider $A \oplus M$ as a ring with multiplication

$$(a, m)(a', m') = (aa', am' + a'm)$$

Denote this ring by $A \oplus_0 M$. Show that there is a one to one correspondence between $\mathrm{Der}_R(A, M)$ and morphisms of R-algebras morphisms

$$A \to A \oplus_0 M$$

that are sections of the projection $A \oplus_0 M \to A$. Why these maps are all uninteresting if we forget about nilpotents?

(3) Show that $A[\epsilon] := A[t]/t^2$ is isomorphic to $A \oplus_0 A$.

(4) Let $X \to \mathrm{Spec}(k)$ a scheme over a field k. Let $x : \mathrm{Spec}(k) \to X$ be a k-rational point. We see $k(x)$ as an $\mathcal{O}_{X,x}$-algebra by the quotient map. Show that there are identifications

$$\mathrm{Der}_k(\mathcal{O}_{X,x}, k(x)) \cong \mathrm{Vect}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)) \cong \mathrm{Sch}_{k,x}(\mathrm{Spec}(k[\epsilon], X))$$

where $\mathrm{Sch}_{k,x}(\mathrm{Spec}(k[\epsilon], X))$ denotes k-schemes maps[2] that sends the point of $\mathrm{Spec}(k[\epsilon])$ to x

Exercise to hand in. *Trivialization of the tangent bundle of an elliptic curve.* (Due Wednesday November 5, 12:00) Please write your solution in TEX.

(1) Let R be a ring, $P = R[x_1, \dots, x_n]$ and $P \to A$ a surjection, with kernel I. Show that there is an exact sequence (understand and make explicit the first map in the claimed sequence)

$$0 \to \mathrm{Der}_R(A, A) \to \bigoplus_{i=1}^{n} A \frac{\partial}{\partial x_i} \to \mathrm{Hom}_A(I/I^2, A)$$

where the last map sends $\frac{\partial}{\partial x_i}$ to the map induced by the composition

$$\frac{\partial}{\partial x_i} : I \to R[x_1, \dots, x_n] \to A$$

Why does it pass to the quotient by I^2?

We denote by $\mathrm{T}^1_{A|R} = \mathrm{Der}_R(A, A)$, the *$A$-module of R- derivations of A*. We also call this module the *tangent bundle of A over R.*

(2) Let

$$E = \mathrm{Proj}\left(\frac{\mathbb{C}[X, Y, Z]}{(Y^2Z - (X^3 + Z^3))}\right).$$

Denote by x, y the images of $\frac{X}{Z}, \frac{Y}{Z}$ in $A_Z := \mathcal{O}_E(D_+(Z))$ and s, t the images of $\frac{X}{Y}, \frac{Z}{Y}$ in $A_Y := \mathcal{O}_E(D_+(Y))$. Show using the exact

[2] meaning that the composition $\mathrm{Spec}(k[\epsilon]) \to X \to \mathrm{Spec}(k)$ is the one associated to $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ being the inclusion.

sequence from above that (meaning that any derivation is a scalar multiplication of the written generator)

$$\mathrm{T}^1_{A_Z|\mathbb{C}} = A_Z(2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y}) \quad \mathrm{T}^1_{A_Y|\mathbb{C}} = A_Y((3t^2-1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t}).$$

(3) Moreover show that the generators displayed above agree on the intersection $D_+(YZ)$, giving a non-vanishing global section π of $\mathrm{T}_{E|\mathbb{C}}$ the sheaf defined by gluing by $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(U), \mathcal{O}(U))$ for any open affine U[3] implying that

$$\mathrm{T}_{E|\mathbb{C}} = \mathcal{O}_{E|\mathbb{C}}\pi.$$

[3]You can assume that this well defined so that this sheaf indeed glues, so you only need to verify that these derivations agree on the intersection.