

Exercises – week 5

Exercise 1 and 2 are meant to prove Chevalley's theorem.

Exercise 1. *Projection from affine spaces.* Let R be a ring.

(1) Show that

$$\pi: \operatorname{Spec}(R[t]) \rightarrow \operatorname{Spec}(R)$$

is open. More precisely, if $f = \sum a_i t^i$ show that

$$\pi(D(f(t))) = \bigcup_i D(a_i).$$

(2) Let $g(t) \in R[t]$ be a monic polynomial and $f(t) \in R[t]$. Remark that $R[t]/g(t)$ is a free R -module of rank $\deg(g)$. Let $\chi(X) = \sum_i^n r_i X^i$ be the characteristic polynomial of the multiplication by $f(t)$ on $R[t]/g(t)$. Show that

$$\pi(D(f) \cap V(g)) = \bigcup_i^{n-1} D(r_i).$$

Exercise 2. *Chevalley's theorem.* Let X be a Noetherian topological space. A subset $T \subset X$ is called *constructible* if it can be written as a finite union of sets of the form $U \cap V^c$ where U and V are open sets.

(1) Show that if $X = \operatorname{Spec}(R)$ for a Noetherian ring R , a subset is constructible if and only if it can be written as a finite union of subsets of the form $D(f) \cap V(g_1, \dots, g_m)$ with $f, g_1, \dots, g_m \in R$.

(2) Show using the above exercise that

$$\pi: \operatorname{Spec}(R[t]) \rightarrow \operatorname{Spec}(R)$$

sends constructible subsets to constructible subsets.

Hint: Show by induction on $\sum_i \deg(g_i)$ that if $f, g_1, \dots, g_m \in R[t]$ are polynomials, the image of $D(f) \cap V(g_1, \dots, g_m)$ is constructible. To conduct the induction step, consider α the leading coefficient of g_1 . Break down the study on the open and closed $D(\alpha)$ and $V(\alpha)$ to reduce the sum of the degrees.

(3) Deduce *Chevalley's theorem*. Let $f: X \rightarrow Y$ be a finite type morphism between Noetherian schemes. Then f sends constructible subsets to constructible subsets.¹

Remark. In general the topological image of a morphism of schemes can fail to be open or closed but in cases where Chevalley's theorem applies, it

¹The generalization to non-Noetherian settings requires more careful definitions, but once these definitions are addressed the proof is the same.

tells that it is still not too far from it and manageable. In particular one can endow the image with a scheme structure.

Exercise 3. *An application of Chevalley's theorem.* Let $f: X \rightarrow Y$ be a finite type dominant map between Noetherian schemes with Y irreducible. Use Chevalley's theorem to show that the topological image $f(X)$ contains an open set.

Exercise 4. *Nullstellensatz via Chevalley.* Let k be a field and \mathfrak{m} be a maximal ideal of $k[x_1, \dots, x_n]$.

- (1) Show by contradiction that $\mathfrak{p}_i = k[x_i] \cap \mathfrak{m}$ is maximal (so $\neq 0$) for each $i = 1, \dots, n$.

Hint: If $\mathfrak{p}_i = 0$, we have a dominant map $\text{Spec}(k(\mathfrak{m})) \rightarrow \mathbb{A}_k^1$.

The above is called *Zariski's lemma* and is the key to Nullstellensatz. Deduce from the lemma proved in item (1) the following direct consequences.

- (2) Deduce the *Nullstellensatz*, meaning that $k[x_1, \dots, x_n]/\mathfrak{m}$ is a finite field extension of k , and that

$$\mathfrak{m} = (\mathfrak{p}_1, \dots, \mathfrak{p}_n).$$

- (3) Let $A \rightarrow B$ a k -algebra map between finite type k -algebras. Show that $f: \text{Spec}(B) \rightarrow \text{Spec}(A)$ carries closed points to closed points.
- (4) Deduce that any finite type finite type k -algebra A is *Jacobson*, meaning that the nilradical (intersection of all primes, see week 3 exercise 1) of A is equal to the intersection of maximal ideals of A .
- Hint: for f not nilpotent, use the preceding point with $A \rightarrow A_f$.*

Exercise 5. *Integrality/reducedness of Proj.* Let B be an \mathbb{N} -graded integral/reduced ring. Show that $\text{Proj}(B)$ is an integral/reduced scheme.

Exercise 6. *Fibers.*

- (1) Compute the fibers of the morphism

$$\text{Spec}(\mathbb{Z}[x, y, z]/(2zx + 9y^2)) \rightarrow \text{Spec}(\mathbb{Z}).$$

Which fiber is reduced? Which fiber is integral?

- (2) Compute the fibers of the morphism, where p is a prime number

$$\text{Spec}(\mathbb{Z}[x, y]/(xy^2 + p)) \rightarrow \text{Spec}(\mathbb{Z}).$$

Which fiber is reduced? Which fiber is integral?

Exercise 7. *Fibers (2).*

- (1) Show that for the morphism $\text{Spec}(k[x, y]/(xy)) \rightarrow \text{Spec}(k[x])$, induced by the obvious map $k[x] \rightarrow k[x, y]/(xy)$, every fiber is irreducible, although the $\text{Spec}(k[x, y]/(xy))$ is not.
- (2) Show that for the morphism $\text{Spec}(\mathbb{Q}[t]) \rightarrow \text{Spec}(\mathbb{Q}[t])$ induced by $t \mapsto t^2$ there are infinitely many closed points with irreducible fibers and infinitely many closed points with non-irreducible fibers.