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AG II - Schemes and sheaves

Exercises – week 4

Some notation. We introduce some notation needed for exercise 1 below. Let A and B be $\mathbb{N}$-graded rings. Let $d \geq 1$. We define the graded ring

$$A^{(d)} = \bigoplus_{n \geq 0} A_{nd}$$

and $v_d \colon A^{(d)} \to A$ for the canonical inclusion as a subring. This subring is called the d-Veronese subring.
We say that a ring map $\psi \colon A \to B$ is *homogeneous of degree* d if for $n \geq 0$ A_n maps to B_{dn}[1]. For example for the usual grading on $\mathbb{Z}[x]$ show that $x \mapsto x^d$ is homogeneous of degree d. Also, v_d is homogeneous of degree d.

Exercise 1. *Functoriality of* Proj. Let A and B be $\mathbb{N}$-graded rings. Let $\psi \colon A \to B$ be an homegeneous map of degree d for some $d \geq 1$.
To the contrary of Spec, the functoriality of Proj is not evident. The reason is that $\psi^{-1}(\mathfrak{p})$ for a prime $\mathfrak{p} \in \mathrm{Proj}(B)$ may contain the irrelevant ideal A_+.

(1) Show that
$$U(\psi) = \{\mathfrak{p} \in \mathrm{Proj}(B) \mid \psi(A_+) \not\subset \mathfrak{p}\}$$
is open. Namely show that it is the union of opens $D_+(\psi(f))$ for all homogeneous $f \in A_+$.
(2) Find an example where $U(\psi)$ is a non-empty open strict subspace of $\mathrm{Proj}(B)$.
(3) Show that ψ^{-1} defines a map of schemes $r_\psi \colon U(\psi) \to \mathrm{Proj}(A)$. Do this by defining a map $D_+(\psi((f)) \to D_+(f)$ for all homogeneous $f \in A_+$ and then glue.
(4) Show that if there exists a k_0 such that for all $k \geq k_0$ the map $A_k \to B_{dk}$ is surjective then $U(\psi) = \mathrm{Proj}(B)$. Show moreover that in this case r_ψ is a topological closed embedding with image $V_+(\ker(\psi))$.
(5) Show that if there exists a k_0 such that for all $k \geq k_0$ the map $A_k \to B_{dk}$ is an isomorphism then r_ψ is an isomorphism.
(6) Deduce that for any $d \geq 1$ and for $v_d \colon A^{(d)} \to A$ the map r_{v_d} is an isomorphism.

Exercise 2. *Basic properties of* $\mathbb{P}^n_A$.
Let A be a ring and $A[x_0, \ldots, x_n]$ the polynomial ring in $n+1$ variables over A. We define $\mathbb{P}^n_A \coloneqq \mathrm{Proj}(A[x_0, \ldots, x_n])$.

(1) Show that $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$.

[1] Note that this means that the map factors through the d-Veronese subring.

Note: this is a first instance of a more general fact about projective varieties and can be thought of as an algebraic instance of the maximum modulus principle in complex analysis. See (3).

(2) Assume that $A = k$, where k is an algebraically closed field. Show that the closed points of $\mathbb{P}^n_k$ are identified with $(n+1)$-tuples $[a_0 : \ldots : a_n]$ satisfying the following properties:
   - $a_i \in k$ for all i,
   - not all a_i are 0, and
   - two $(n+1)$-tuples $[a_0 : \ldots : a_n]$ and $[b_0 : \ldots : b_n]$ are identified if there exists $c \in k^*$ such that $b_i = c \cdot a_i$ for all i.

   In other words, the points are identified with $(k^{n+1} \setminus 0)/k^{\times}$, *i.e.* linear subspaces of dimension 1 of k^{n+1}.

(3) Let $A = k$ be a field and B be a k-algebra. Show that every morphism of k-schemes $\mathbb{P}^n_k \to \operatorname{Spec}(B)$ is constant at the level of topological spaces with image a closed point which is k-rational.

(4) Let d be a positive integer and set $m := \binom{n+d}{d} - 1$. Use Exercise 1 and monomials of degree d to define an everywhere defined morphism[2], that we call a d-th Veronese embedding of $\mathbb{P}^n_A$

$$\psi_d \colon \mathbb{P}^n_A \to \mathbb{P}^m_A.$$

(5) Let k be an algebraically closed field. Describe the image of a second Veronese embedding $\psi_2 \colon \mathbb{P}^1_k \to \mathbb{P}^2_k$. Furthermore, using part (2), describe the closed points of the image as triples $[a_0 : a_1 : a_2]$.

Exercise 3. *Finite covers of* $\mathbb{P}^1_{\mathbb{C}}$. Let $n \geq 1$. Consider the self map c_n of $\mathbb{C}$-schemes on $\operatorname{Proj}(\mathbb{C}[x, y]) = \mathbb{P}^1_{\mathbb{C}}$ induced by Proj from the $\mathbb{C}$-algebra map $x \mapsto x^n$ and $y \mapsto y^n$ on $\mathbb{C}[x, y]$.

(1) Compute the preimage by this map of $D_+(x)$ and $D_+(y)$, show it's affine.

(2) Compute all the fibers of the map.

Exercise 4. *Ramifications of a self map of* $\mathbb{P}^1$.

- We say that a map of schemes $f \colon X \to Y$ is *finite locally free* if there is a covering of Y by open affines $\operatorname{Spec}(A_i)$, with affine preimage $\operatorname{Spec}(B_i)$, such that induced map $A_i \to B_i$ turns B_i into a finite free A_i-module. When for every i the dimension of B_i is the same, say d, we say that the map is *finite locally free of degree d.*
- We say that a finite locally free map $X \to Y$ is *ramified at* $y \in Y$ if the geometric fiber $X_{\overline{y}}$ is not reduced.

(1) Show that the self map c_n from exercise 3 is finite locally free of degree n and identify it's ramification points.

(2) Let R be a ring. Show that the map induced on $\mathbb{P}^1_R = \operatorname{Proj}(R[x, y])$ by the R-algebra self map $x \mapsto ax + by$ and $y \mapsto cx + dy$ is an automorphism if $ad - bc \in R^{\times}$. We denote this map $m_{(a,b,c,d)}$.

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[2] Induce a map using Proj from an homogeneous map of degree d that sends monomials of degree 1 to all monomials of degree d.

If $R = \mathbb{C}$ and if we identify $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C}) = \mathbb{C} \cup \infty$, how is this map expressed on $\mathbb{C}$-rational points?

(3) Consider the composition

$$\mathbb{P}^1_{\mathbb{C}} \xrightarrow{c_n} \mathbb{P}^1_{\mathbb{C}} \xrightarrow{m_{(1,-1,1,1)}} \mathbb{P}^1_{\mathbb{C}} \xrightarrow{c_2} \mathbb{P}^1_{\mathbb{C}}.$$

Show it's finite locally free of fixed degree. What is the degree? What are the ramification points? Compute scheme theoretic fibers at all ramification points.

Exercise to hand in. *A family of curves of* Spec($\mathbb{Z}$). (Due Wednesday October 15, 12:00) Please write your solution in TEX.
Consider the closed sub-scheme of $\mathbb{P}^2_{\mathbb{Z}}$

$$C := V_+(X_0^2 + 5X_1^2 + 7X_2^2) = \operatorname{Proj}\left(\frac{\mathbb{Z}[X_0, X_1, X_2]}{(X_0^2 + 5X_1^2 + 7X_2^2)}\right) \subset \mathbb{P}^2_{\mathbb{Z}}.$$

For a prime number $p \in \mathbb{N}$, we denote by C_p the fiber of C at $(p) \in \operatorname{Spec}(\mathbb{Z})$.

(1) For which primes C_p is reduced?
(2) For which primes C_p is *geometrically* reduced?[3]
(3) Compute $\mathbb{F}_3$-rational points and $\mathbb{F}_5$-rational points of C_3 and C_5 respectively.
(4) We denote by $C_{\mathbb{Q}} = C \times \operatorname{Spec}(\mathbb{Q})$ the *generic fiber* of C. Show that $C_{\mathbb{Q}}$ has no $\mathbb{Q}$-rational points.

[3]This means that $C \times \operatorname{Spec}(\overline{\mathbb{F}}_p)$ is reduced.