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AG II - Schemes and sheaves

Exercises – week 14

Exercise 1. *Conics.* Let k be a field. Let $C = V_+(F) \subset \mathbb{P}^2_k$ for F a degree 2 homogeneous polynomial. Suppose that C is *geometrically integral*, meaning that $C_{\overline{k}}$ is integral.

(1) Let $\nu\colon C' \to C_{\overline{k}}$ the normalization of $C_{\overline{k}}$. Consider the exact sequence
$$0 \to \mathcal{O}_{C_{\overline{k}}} \to \nu_*\mathcal{O}_{C'} \to K \to 0$$
where K is defined to be the cokernel. Using the long exact sequence in cohomology, show that $K = 0$, and deduce that $C_{\overline{k}}$ is regular.[1]

(2) Show that $C(k) \neq \emptyset$ if and only if $C \cong \mathbb{P}^1_k$.

(3) Consider the closed subscheme of $\mathbb{P}^2_{\mathbb{Z}}$
$$C := V_+(X_0^2 + 5X_1^2 + 7X_2^2) = \operatorname{Proj}\left(\frac{\mathbb{Z}[X_0, X_1, X_2]}{(X_0^2 + 5X_1^2 + 7X_2^2)}\right) \subset \mathbb{P}^2_{\mathbb{Z}}.$$
For which prime numbers p do we have $C_p \cong \mathbb{P}^1_{\mathbb{F}_p}$?

Exercise 2. *Complements of ample divisors are affine.* Let k be a field. Let $X \to \operatorname{Spec}(k)$ be a projective k-scheme. Let D be an ample effective Cartier divisor. Show that $X \setminus D$ is affine.

Exercise 3. *Curves.* Let k be an algebraically closed field. We fix C a smooth connected projective curve over k. We let[2]
$$g := h^1(C, \mathcal{O}_C)$$
the *genus* of the curve C.

(1) *Riemann-Roch.* Show that for any $\mathcal{L} \in \operatorname{Pic}(C) \cong \operatorname{Cl}(C)$ we have[3]
$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g.$$
Redo the proof if you have seen it in class.

(2) Let $x \in C(k)$. Show that there exists a rational function $f \in K(C)$ regular everywhere except at x.

(3) Let U be a non-empty open. Show that there is a rational function regular only at points of U.

(4) Suppose that U is a *strict open* of C. Construct a morphism $f\colon C \to \mathbb{P}^1_k$ with $f^{-1}(D_+(x_0)) = U$.

[1] This implies that $C_{\overline{k}}$ is smooth over $\overline{k}$, implying that C is smooth over k. In particular, C is regular.

[2] For a coherent sheaf $\mathcal{F} \in \operatorname{Coh}(C)$ we denote $h^i(C, \mathcal{F}) = \dim_k(H^i(C, \mathcal{F}))$.

[3] The degree of a line bundle is the degree of the corresponding class of divisors in the class group.

(5) Let $Z := C \setminus U$. Show that there is an effective divisor D with support exactly Z with D being ample. Deduce that any strict open in an integral regular proper curve is affine.
(6) Let $C' \to C$ be a dominant map between smooth connected proper k-curves. Show that the map is affine, and finite[4] flat of degree $[K(C') \colon K(C)]$.
(7) Show that if there is a closed point $P \in C(k)$ with $\mathrm{H}^0(C, \mathcal{O}_C(P)) = 2$, then $C \cong \mathbb{P}^1_k$ over k.
(8) Show that for any effective divisor D, we have $h^0(C, \mathcal{O}(D)) \leq \deg(D)+1$. Show that equality happens if and only if $D = 0$ or $g = 0$.

Exercise 4. *More on elliptic curves.*
Let E be a smooth connected projective curve over an algebraically closed field k. Suppose that

$$h^1(E, \mathcal{O}_E) = 1.$$

(1) Using Serre duality and Riemann-Roch, show that $\Omega_{E|k} \cong \mathcal{O}_E$.
(2) *Weierstrass equation.* Fix a k-rational point $e \in E(k)$. Such a pair (E, e) is called an *elliptic curve.* Suppose now that the characteristic of the field is not 2 or 3. Show that one can find $x \in H^0(E, \mathcal{O}(2e)) \setminus k$, $y \in H^0(E, \mathcal{O}(3e)) \setminus H^0(E, \mathcal{O}(2e))$ and coefficients $a, b \in k$ with

$$y^2 = x^3 + ax + b.$$

(3) *Weierstrass embedding.* Using that $\mathcal{O}(3e)$ is very ample, deduce that there is a closed embedding in $\mathbb{P}^2_k$

$$E \to V_+(Y^2 Z = X^3 + aXZ^2 + bZ^3) \subset \mathbb{P}^2_k.$$

What is the image of e in coordinates?

Exercise 5. *Cohomology and affine maps.* Let E be an elliptic curve embedded in $\mathbb{P}^2_k$ as in Exercise 4. Consider the partially defined projection $\mathbb{P}^2_k \dashrightarrow \mathbb{P}^1_k$ on the first two components. Show that there is a unique induced map $f \colon E \to \mathbb{P}^1_k$ and compute the cohomology of $f_*\mathcal{O}(ne)$ for $n \in \mathbb{Z}$.

Exercise 6. *Projection formula.* Let $f \colon X \to Y$ be a morphism of ringed spaces and let $\mathcal{E}$ be a locally free sheaf of finite rank on Y. Let $\mathcal{F}$ be any sheaf of $\mathcal{O}_X$-module. Show that there is a natural isomorphism

$$R^i f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong R^i f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

Exercise 7. *Hodge numbers of projective space.* Let k be a field, and X be proper k-scheme. The *Hodge numbers of X* are defined as

$$h_{p,q}(X) := \dim_k \left(H^q \left(X, \Omega^p_{X|k} \right) \right).$$

[4]Once you get that the map is affine, finiteness is still to show, This does not follow immediately from the fact that it is finite at the generic fiber. Properness of the map helps. See the extra exercise at the end of the sheet.

These are important invariants of X from the view point of algebraic geometry. Using the Euler sequence and the remark below, show that for $0 \leq p, q \leq n$

$$h_{p,q}(\mathbb{P}^n_k) = \begin{cases} 1 \text{ if } p = q \\ 0 \text{ otherwise.} \end{cases}$$

Remark. Let

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

an exact sequence of finite locally free sheaves on a scheme X. Then for any $n \in \mathbb{N}$, there is an induced filtration

$$\bigwedge^n \mathcal{E} = F^0 \supset F^1 \supset \cdots \supset F^n \supset F^{n+1} = 0.$$

such that for every $0 \leq i \leq n$ we have an induced exact sequence

$$0 \to F^{i+1} \to F^i \to \bigwedge^i \mathcal{E}' \otimes \bigwedge^{n-i} \mathcal{E}'' \to 0.$$

Extra exercise. *Integral and finite maps.* Let $A \to B$ be a ring map. Suppose that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is universally closed. The goal is to show that $A \to B$ is integral. (The converse holds using the going-down theorem.)

(1) Show that $A \to B$ is integral if and only for every $b \in B$ the kernel J_b of the composition

$$A[t] \to B[t] \xrightarrow{\mathrm{ev}_{b^{-1}}} B_b$$

contains a polynomial with constant coefficient 1.

(2) Show that J_b contains a polynomial with constant coefficient 1 if and only if $\mathrm{Spec}(A[t]/(J_b + (t)))$ is empty.

(3) Using that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is universally closed, show that the induced map $\mathrm{Spec}(B_b) \to \mathrm{Spec}(A[t]/J_b)$ from item (1) is surjective. *Hint: This is the image of $V(bt-1)$ under the map* $\mathrm{Spec}(B[t]) \to \mathrm{Spec}(A[t])$.

(4) Argue that the squares in the diagram below are pullbacks and conclude.

$$\begin{array}{ccccc} \emptyset & \longrightarrow & \mathrm{Spec}(A[t]/(J_b + (t))) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} \\ \mathrm{Spec}(B_b) & \longrightarrow & \mathrm{Spec}(A[t]/J_b) & \longrightarrow & \mathrm{Spec}(A[t]) \end{array}$$

Now deduce that: *a morphism of schemes $X \to Y$ is finite if and only if it is affine and proper.*