Dr. Domenico Valloni
Dr. Alapan Mukhopadhyay
Léo Navarro Chafloque

EPFL, fall semester 2025
AG II - Schemes and sheaves

Exercises – week 12

Exercise 1. *Very ample divisors.* Let k be a field. Let S be a $\mathbb{N}$-graded ring finitely generated in degree 1 with $S_0 = k$. Denote by $X = \mathrm{Proj}(S)$. Suppose that X is integral and $\mathcal{O}_X(X) = k$.[1]

(1) Show that $\mathcal{O}_X(1)$ is k-very ample.
(2) If $\dim(X) \geq 1$, show that $\mathbb{Z} \xrightarrow{\mathcal{O}_X(1)} \mathrm{Pic}(X)$ is injective.
Hint: If $\mathcal{O}_X(1)$ is torsion, it would imply that $\mathcal{O}_X$ is k-very ample.
(3) If X is normal, deduce that if $0 \neq s \in \mathcal{O}_X(1)(X)$, then $\mathrm{div}(s) \in \mathrm{Cl}(X)$ has infinite order.

Exercise 2. *Homotopy invariance and an instance of Künneth.* Let X be a Noetherian, integral separated and regular in codimension 1 scheme[2]. Let $\pi \colon Y \to X$ be a scheme morphism such that there exists an affine open cover $(U_i = \mathrm{Spec}(A_i))$ of X with $\mathbb{A}^1_{U_i} \cong \pi^{-1}(U_i)$ over U_i.

(1) Follow the steps below to deduce that $\mathrm{Cl}(Y) \cong \mathrm{Cl}(X)$.
   (a) Let t be the image of $t \in A_i[t]$ in $K(Y)$ by an isomorphism $\mathbb{A}^1_{U_i} \cong \pi^{-1}(U_i)$ from the hypothesis for a fixed i. Show that $K(Y) = K(X)(t)$.
   (b) We separate codimension 1 points of Y in two types. Let $y \in Y$ be a point. We say that y is of *type 1* if $\pi(y)$ is of codimension 1. We say that point $y \in Y$ is of *type 2* if $\pi(y)$ is the generic point of X.[3]
      (i) When y is a point of type 1, show that if $\pi(y) = \mathfrak{p} \in \mathrm{Spec}(A_i)$ then y identifies with the prime ideal $\mathfrak{p}A_i[t] \in \mathbb{A}^1_{U_i} = \mathrm{Spec}(A_i[t])$ up to the isomorphism $\mathbb{A}^1_{U_i} \cong \pi^{-1}(U_i)$ from the hypothesis. From this analysis, deduce that $\mathcal{O}_{Y,y}$ is a DVR.
      (ii) Show that points of type 2 are in one to one correspondence with closed points of $\mathbb{A}^1_{K(X)}$. In this case, deduce also that the local rings $\mathcal{O}_{Y,y}$ are DVR's.
   (c) Deduce that Y is also regular in codimension 1. Show also that Y is Noetherian, integral separated.
   (d) Let y be a point of type 2. Show that y is linearly equivalent to a linear combination of points of type 1. More precisely show that if $(f) \in K(X)[t]$ correspond to y, then show that $y - \mathrm{div}(f)$ is of type 1.

[1]This condition follows from previous assumptions if k is algebraically closed.
[2]These hypothesis are the ones to use the notion of *Weil divisors*.
[3]Only these two cases are possible because π is flat and therefore the codimension can only drop as a consequence of going-up.

(e) Let $Z \subset X$ be a prime divisor. Show that $\pi^{-1}(Z)$ the topological preimage of Z is a prime divisor with a generic point of type 1. Moreover show that if D is a Weil divisor of $X = \sum n_i Z_i$ then $\pi^{-1}(D) := \sum n_i \pi^{-1}(Z_i)$ cannot be linearly equivalent to a principal divisor of Y unless Z is already principal in X.

(f) Using the above show that the map

$$\pi^*\colon \operatorname{Cl}(X) \to \operatorname{Cl}(Y)$$

sending $\sum n_i Z_i \mapsto \sum_i n_i \pi^{-1}(Z_i)$ is a well defined isomorphism.

(2) Consider $X \times \mathbb{P}^n$ for $n \geq 1$. Using the exact sequence from Week 10, Exercise 3 with the open $X \times D_+(X_0)$, and point (1) of this exercise deduce that we have a split exact sequence (find the splitting)

$$1 \longrightarrow \mathbb{Z} \longrightarrow \operatorname{Cl}(X \times \mathbb{P}^n) \longrightarrow \operatorname{Cl}(X) \longrightarrow 1$$

so that $\operatorname{Cl}(X \times \mathbb{P}^n) \cong \operatorname{Cl}(X) \times \mathbb{Z}$.

Exercise 3. *Projective Cone.*
Let S be a $\mathbb{N}$-graded ring finitely generated in degree 1 over S_0. Consider the $\mathbb{N}$-graded ring $S[t]$ with elements of S keeping their grading and with t placed in degree 1. We call $\operatorname{Proj}(S[t])$ with this grading the *projective cone.*

(1) Show that there are natural identifications $V_+(t) = \operatorname{Proj}(S)$ and $D_+(t) = \operatorname{Spec}(S)$. Show furthermore that $V_+(S_+)$ (taken in $\operatorname{Proj}(S[t])$) identifies to $V(S_+)$ in $\operatorname{Spec}(S)$. We denote this closed subscheme by v.

(2) Let $s_0, \dots, s_n$ be generators of S in degree 1. Show that $\operatorname{Proj}(S[t]) \setminus v$ is covered by the open sets $D_+(s_i)$ and that each open set is isomorphic to $\operatorname{Spec}(S_{(s_i)}[t])$. Deduce that we have a natural map

$$p\colon \operatorname{Proj}(S[t]) \setminus v \to \operatorname{Proj}(S).$$

(3) Let k be an algebraically closed field, and suppose $S_0 = k$. Suppose that S is integral, Noetherian and normal. Suppose that $X = \operatorname{Proj}(S)$ is of dimension ≥ 1. By Exercise 2, note that p^* induces an isomorphism on class groups. Deduce that, if $C = \operatorname{Spec}(S)$ denotes the cone of X then we have an exact sequence

$$1 \to \mathbb{Z} \to \operatorname{Cl}(X) \to \operatorname{Cl}(C) \to 1$$

where the first morphism sends 1 to the class of $\mathcal{O}_X(1)$, and the second is the composition of p^* and the restriction to C. Use the exact sequence from Week 10, Exercise 3 to this effect.

Exercise 4. *Computations of class groups on quadric hypersurfaces.* Suppose that k is algebraically closed and $char(k) \neq 2$. Let $2 \leq r \leq n$. Consider the ring (equipped with the standard grading)

$$S_r = k[x_0, \dots, x_n]/(x_0^2 + \cdots + x_r^2).$$

You can assume that this ring is normal.

(1) Show that up to a linear change of variable, we can suppose that

$$S_r = k[x_0, \dots, x_n]/(x_0x_1 + x_2^2 + \cdots + x_r^2).$$

Denote by $C_r = \mathrm{Spec}(S_r)$ and $X_r = \mathrm{Proj}(S_r)$.

(2) Show that $\mathrm{Cl}(C_r)$ is cyclic when $r \neq 3$
Hint: Consider the prime[4] *divisor* $V(\sqrt{(x_1)})$ *and an exact sequence of class groups.*

(3) Show that $\mathrm{Cl}(C_2) \cong \mathbb{Z}/2\mathbb{Z}$.
Hint: Consider the same exact sequence. Show that $Y = V(\sqrt{(x_1)}) = V(x_1, x_2)$ *is not principal and that* $2Y = 0$.

(4) Show that $\mathrm{Cl}(C_3) \cong \mathbb{Z}$.
Hint: show that after a suitable change of variable we see that $X_r \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$. *Then use Exercise 2.(2) and Exercise 3.(3)*

(5) Show that $\mathrm{Cl}(C_r) \cong 0$ of $r \geq 4$. In particular, S_r is factorial.
Hint: show that (x_1) *is prime in this case and conclude.*

(6) Use the exact sequence of the last point of the above exercise to compute $\mathrm{Cl}(X_r)$ for all $r \geq 2$.

[4]That's where $r = 3$ is used, so that this ideal is prime.

Exercise to hand in. *Line bundles on cubics.* (Due Wednesday December 17, 12:00) Please write your solution in $\TeX$.
Let k be an algebraically closed field and C be a smooth projective curve over k, *i.e.* a one dimensional integral, regular and projective scheme over k.

(1) For a nonzero rational function $f \in K(C)$, consider a nonempty open subset $U \subseteq C$ such that $f \in \Gamma(U, \mathcal{O}_C)$. Argue that the induced map of k-schemes $f : U \to \mathbb{A}^1_k \subseteq \mathbb{P}^1_k$, extends to a map $\tilde{f} : C \to \mathbb{P}^1_k$.
(2) Prove that $\tilde{f}^*([0:1] - [1:0]) = \mathrm{div}(f)$. Conclude that if there are two distinct k-points P, Q on C such that $P \sim Q$ as Weil divisors, then C must be isomorphic to $\mathbb{P}^1_k$.
(**Hint:** Use part 3, Ex 3, Week 11)
(3) Recall that
$$\tilde{f} \colon C \to \mathbb{P}^1_k$$
corresponds to collection of global sections on an invertible sheaf on C. Can you identify the invertible sheaf and the corresponding global sections in terms of the rational function f?

Now assume that C is defined by a cubic polynomial in $\mathbb{P}^2_k$. In this and the following exercise, you can assume that for any degree one Weil divisor D, $\mathcal{O}_C(D)$ has at least one nonzero global section.

(4) Prove that for any degree one divisor D, there is a unique point $P \in C$, such that $D \sim P$. Now fix $P \in C$. Prove that for any degree zero divisor D, there is a unique point $Q \in C$ such that $D \sim Q - P$.
Rmk: So $Q \mapsto Q - P$ is a bijection between $C(k)$ and degree zero divisors in $\mathrm{Cl}(C)$. So the group structure on $\mathrm{Cl}(C)$ induces a group structure on $C(k)$. This is the group structure on the Elliptic curve with base point P defined by this cubic equation.
(5) Let C be as in the previous part, fix $P \in C$. Using (4), deduce that the following sequence is exact
$$0 \longrightarrow C(k) \longrightarrow \mathrm{Cl}(C) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$
where the left arrow sends Q to $Q - P$. Conclude that $\mathrm{Pic}(C) \cong C(k) \oplus \mathbb{Z}$.