

Exercises – week 11

Exercise 1. *Effective Cartier divisors.* Let X be an integral scheme. A Cartier divisor on X represented by (f_i, U_i) is said to be *effective* if $f_i \in \mathcal{O}(U_i)$ for every i .

- (1) Show, by looking at the ideal sheaf generated by the f_i 's, that effective Cartier divisors are in one-to-one correspondence with ideal sheaves \mathcal{I} that are a locally free sheaves of rank 1. We take this point of view in what follows.
- (2) Let \mathcal{L} be a locally free sheaf of rank 1. Show that $s: \mathcal{O}_X \rightarrow \mathcal{L}$ is non-zero if and only if the evaluation $\text{ev}_s: \mathcal{L}^\vee \rightarrow \mathcal{O}_X$, defined by $\mathcal{L}^\vee(U) = \text{Hom}_{\mathcal{O}_U}(\mathcal{L}_U, \mathcal{O}_U) \ni \varphi \mapsto \varphi(s)$ is injective.
- (3) Fix a locally free sheaf \mathcal{L} of rank 1. Deduce the following bijection,

$$\frac{(\Gamma(X, \mathcal{L}) \setminus \{0\})}{\mathcal{O}_X(X)^\times} \rightarrow \{\text{Effective Cartier divisors } \mathcal{I} \text{ on } X \text{ with } \mathcal{I} \cong \mathcal{L}^\vee\}$$

that sends the class of a section s to $\text{Im}(\text{ev}_s)$.

- (4) Suppose that $\mathcal{O}_X(X)$ is a field. Show that if \mathcal{L} is a locally free sheaf of rank 1 such that \mathcal{L} and \mathcal{L}^\vee have a non zero section, then $\mathcal{L} \cong \mathcal{O}_X$. *Hint: in this case both \mathcal{L} and \mathcal{L}^\vee correspond to effective Cartier divisors.*
- (5) Additionally assume that X is normal, Noetherian and integral. *Two Weil divisors are called linearly equivalent if their difference is the divisor of some rational function.* Let D be a Weil divisor on X . Show that map sending $f \in \Gamma(X, \mathcal{O}_X(D))$ to $\text{div}(f) + D$ gives a one to one correspondence

$$\frac{\Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}}{\mathcal{O}_X(X)^\times} \rightarrow \{\text{Effective Weil divisors linearly equivalent to } D\}.$$

Careful, hypothesis does not imply that $\mathcal{O}_X(D)$ defined as

$$\mathcal{O}_X(D)(U) = \{g \in K(X) \mid g \neq 0, (\text{div}(g) + D) \cap U \text{ is effective}\}$$

is a line bundle. So you have to prove it independently of item (3).

Remark. Let X be a projective k -scheme where k is algebraically closed. We will show later in the course that $\Gamma(X, \mathcal{O}_X) = k$ as in the case of the projective space. Using (3) of Exercise 1, we see, for \mathcal{L} a line bundle on X , that

$$\mathbb{P}(\Gamma(X, \mathcal{L}))(k) = \{\text{Effective Cartier divisors } \mathcal{I} \text{ on } X \text{ with } \mathcal{I} \cong \mathcal{L}^\vee\}$$

Therefore $\mathbb{P}(\Gamma(X, \mathcal{L}))$ gives a natural k -scheme structure on the set on the right hand side. This set is called *the complete linear system of \mathcal{L}* .

Exercise 2. *Extension principle for normal schemes.* Let k be a field. Let $X \rightarrow \text{Spec}(k)$ be a normal and integral k -scheme of finite type and let $Y \rightarrow \text{Spec}(k)$ a proper k -scheme. Let $U \subset X$ be an open subscheme. Say we have

$$f: U \rightarrow Y$$

a morphism of k -schemes. Show that we can extend f to a map $g: V \rightarrow Y$ where V is an open of X containing all generic points of codimension 1 irreducible closed subschemes (=Weil Divisors).

Remark. This generalize the *extension principle* from Exercise 1, Week 7. The proof is very similar!

Exercise 3. *Divisors on regular curves.* Let k be an algebraically closed field. We say that C is a *regular k -curve over k* is a one dimensional separated, integral and regular scheme over k . Weil (=Cartier in this case) divisors are then of the form

$$D = \sum_i n_i x_i$$

for x_i being closed points of C . We define the *degree* of a divisor $D = \sum_i n_i x_i$ to be

$$\deg(D) = \sum_i n_i \in \mathbb{Z}.$$

Let $f: C' \rightarrow C$ a finite k -morphism between regular k -curves. We define the *pullback* of an irreducible divisor (=closed point)

$$f^*x = \sum_{y \in C'_{cl} \text{ s.t. } x=f(y)} v_y(f^\sharp(t_x))y.$$

where f^\sharp denotes the induced map at the local ring. Here, t_x denotes a generator of \mathfrak{m}_x – this well defined because the choice of a generator is up to a unit. We extend f^* by linearity to $\text{Div}(C)$.

- (1) Show that the pullback of a principal divisor is principal, implying that f^* factors through

$$f^*: \text{Cl}(C) \rightarrow \text{Cl}(C').$$

- (2) Show that if the degree of the map (= $[K(C'): K(C)]$) is d , then $\deg(f^*D) = d \deg(D)$. *Hint: it suffices to show the claim for $D = x$ a closed point by linearity.*
- (3) Assume now that C is also proper. Using the extension principle from Exercise 1, Week 7 (or exercise 2 above) show that for every $t \in K(C) \setminus k$ we have a map $f_t: C \rightarrow \mathbb{P}_k^1$ from the inclusion $k(t) \subset K(C)$ such that $f_t^*(0-\infty) = \text{div}(t)$ where 0 denotes $V(t)$ in $\text{Spec}(k[t]) \subset \mathbb{P}_k^1$ and ∞ denotes $V(1/t)$ in $\text{Spec}(k[1/t]) \subset \mathbb{P}_k^1$. deduce that $\deg((t)) = 0$, and that therefore \deg factor through

$$\deg: \text{Cl}(C) \rightarrow \mathbb{Z}.$$

Exercise 4. *Closed subschemes of Proj.* Let R be a graded ring finitely generated over R_0 in degree 1. Let $P = R_0[x_0, \dots, x_n]$ be the graded polynomial ring where x_i 's are placed in degree 1.

- (1) Show, using the sheaf property on the cover $(D_+(x_i))_{i=0}^n$ that the natural map

$$P_m \rightarrow \Gamma(\mathbb{P}_{R_0}^n, \mathcal{O}_{\mathbb{P}_{R_0}^n}(m))$$

is an isomorphism for any $m \in \mathbb{Z}$.

- (2) Show that there is a closed immersion

$$\text{Proj}(R) \rightarrow \mathbb{P}_{R_0}^n$$

for some $n \in \mathbb{N}$, which comes from a chosen graded surjection $R[x_0, \dots, x_n] \rightarrow R$. We fix a such for the rest of the exercise.

- (3) Let $Z \rightarrow \mathbb{P}_{R_0}^n$ a closed subscheme given by a quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}_{R_0}^n}$. Define

$$I := \bigoplus_{m \in \mathbb{N}} \Gamma(\mathbb{P}_{R_0}^n, \mathcal{I}(m)).$$

which is an ideal of P using that $\mathcal{I}(m) \subset \mathcal{O}_{\mathbb{P}_{R_0}^n}(m)$ and (1). Show, using that the natural map $\tilde{I} \rightarrow \mathcal{I}$ is an isomorphism (see the lecture) that the natural map

$$\text{Proj}(P/I) \rightarrow \mathbb{P}_{R_0}^n$$

is identified with $Z \rightarrow \mathbb{P}_{R_0}^n$.

- (4) Now say that $Z \rightarrow \text{Proj}(R)$ is a closed-subscheme. Combine (2) and (3) to deduce that there is a graded ideal $J \subset R$ such that

$$\text{Proj}(R/J) \rightarrow \text{Proj}(R)$$

is identified with $Z \rightarrow \text{Proj}(R)$.

Remark. In the end the ideal constructed in the way suggested by the exercise can be described as follows. Recall that any $r \in R_n$ can be seen as a section in $\Gamma(\text{Proj}(R), \mathcal{O}_{\text{Proj}(R)}(n))^1$. Now the homogeneous elements of degree n of the ideal defined as suggested in the above exercise are exactly the elements that becomes zero when pulling back to Z when seen a global section of $\mathcal{O}_{\text{Proj}(R)}(n)$.

Exercise 5. *Segre embedding.* In this exercise we use that sections of line bundles correspond to morphisms to a projective space, see the lecture.

Let k be a ring. Denote the projection of $\mathbb{P}_k^1 \times_k \mathbb{P}_k^1$ to the first and second factor by p_1 and p_2 respectively. View the first and second copy of \mathbb{P}^1 in the product as $\text{Proj}(k[x_0, x_1])$ and $\text{Proj}(k[y_0, y_1])$ respectively. Show that the global sections $p_1^*(x_i) \otimes p_2^*(y_j)$ for $0 \leq i, j \leq 1$ of $p_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$ give a closed embedding of $\mathbb{P}_k^1 \times_k \mathbb{P}_k^1$ in \mathbb{P}_k^3 .

¹There is always the natural map $R_n \rightarrow \Gamma(\text{Proj}(R), \mathcal{O}_{\text{Proj}(R)}(n))$ which might not be an isomorphism. But using this map we can consider elements of R_n as sections of $\mathcal{O}_{\text{Proj}(R)}(n)$.

Exercise 6. *Fibre dimension (of coherent sheaves).*

Let X be a Noetherian scheme and \mathcal{F} a coherent sheaf on X . Let

$$\varphi: X \rightarrow \mathbb{N}$$

be defined as $\varphi(x) = \dim_{k(x)}(\mathcal{F} \otimes_{\mathcal{O}_X} k(x))$.

Nakayama's lemma may be useful for the following.

- (1) Show that φ is upper semi-continuous meaning that for any $n \geq 0$

$$\{x \in X \mid \varphi(x) \geq n\}$$

is closed.

- (2) If \mathcal{F} is locally free and X connected show that φ is constant.
(3) Show that if X is reduced and connected show that \mathcal{F} is locally free if and only φ is constant.