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AG II - Schemes and sheaves

Exercises – week 10

Exercise 1. *Line bundles and exact sequences.* Let X be a scheme and $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \mathrm{Qcoh}(X)$. Let $\mathcal{L}$ be locally free sheaf of rank 1, *i.e.* a *line bundle.*

(1) Show that
$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
is exact if and only if
$$0 \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{L} \to 0$$
is.
Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ be finite locally free sheaves. Show that if
$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$
is exact if and only if
$$0 \to \mathcal{E}_3^\vee \to \mathcal{E}_2^\vee \to \mathcal{E}_1^\vee \to 0$$
is.

Let $\mathcal{E}$ be a finite locally free sheaf of constant rank n. We define
$$\det(\mathcal{E}) := \bigwedge^n \mathcal{E}.$$

(3) Show that $\det(\mathcal{E})$ is a line bundle. Namely show that $\det(R^{\oplus n}) \cong R$ where R is a ring.
(4) Let $\varphi\colon R^{\oplus n} \to R^{\oplus n}$ be an R-module map, where R is a ring. Show that φ induces a morphism
$$\det(\varphi)\colon \det(R^{\oplus n}) \to \det(R^{\oplus n})$$
which is given by the multiplication by the determinant of the matrix defining φ.
(5) Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ be finite locally free sheaves of constant rank. Suppose that
$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$
is an exact sequence. Then, show that there is an induced isomorphism
$$\det(\mathcal{E}_2) \cong \det(\mathcal{E}_1) \otimes \det(\mathcal{E}_3).$$

Exercise 2. *Divisors that are not Cartier.* Let k be a field and $X = V(xy - zw)$ in $\mathbb{A}^4_k$. Note that X is integral and regular in codimension 1.

(1) Show that the closed subsets in X defined by $x = z = 0$ and $x = w = 0$ are prime divisors that are not Cartier. Denote by D_z and D_w these divisors.

(2) Show that $D_z + D_w$ is a Cartier divisor.

Exercise 3. *Exact sequence for class groups.* Let X be an integral separated scheme which is regular in codimension 1. Let Z be a proper closed subset of X and $U = X \setminus Z$.

(1) Show that $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ defined by $\sum n_i D_i \mapsto \sum n_i (D_i \cap U)$ is surjective.
(2) If $\mathrm{codim}(Z, X) \leq 2$, show that that this map is also injective.
(3) If $\mathrm{codim}(Z, X) = 1$ and Z is irreducible, show that there is an exact sequence
$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 1$$
where $\mathbb{Z} \to \mathrm{Cl}(X)$ send 1 to Z.
(4) Let k be a field. Let Z be the zero set of an irreducible homogeneous polynomial of degree d in $\mathbb{P}^n_k$. Deduce that $\mathrm{Cl}(\mathbb{P}^n_k \setminus Z) \cong \mathbb{Z}/d\mathbb{Z}$.

Exercise 4. *Extension of coherent sheaves.* The goal is to show that if X is a Noetherian scheme, U an open subset and $\mathcal{F}$ is a coherent sheaf on U, then there is a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{G}_{|U} \cong \mathcal{F}$.

(1) Show that on a Noetherian scheme X and $\mathcal{F}$ coherent sheaf, then if
$$\sum_i \mathcal{F}_i = \mathcal{F}$$
where $(\mathcal{F}_i)_{i \in I}$ are sub-coherent sheaves, then there exist a finite refinement $J \subset I$ such that $\sum_{j \in J} \mathcal{F}_j = \mathcal{F}$.
(2) Show that on a Noetherian affine scheme, every quasi-coherent sheaf is the direct colimit of it's coherent sub-sheaves. *Hint: Use the equivalence of categories with modules on global sections.*
(3) Let X be affine and $\iota \colon U \to X$ be an open subcheme. Show the claim in this case. *Hint: Show that $\iota_* \mathcal{F}$ is quasi-coherent, and then use a combination of (1) and (2) to conclude.*
(4) Show the claim in the general case of the statement of the exercise by induction on the number of open affines that are required to cover X. (Being covered by one open affine being the base case of the induction, and is the previous point. The rest is an induction play, see Hint.) *Hint: Say $X = X_1 \cup X_2$ where X_1 and X_2 are open subschemes that can be covered by strictly less open affines than X. By induction extend $\mathcal{F}_{X_1 \cap U}$ to a coherent sheaf $\mathcal{G}_1$ defined on X_1. By gluing $\mathcal{F}$ and $\mathcal{G}_1$ it defines a coherent sheaf $\mathcal{G}'$ defined on $X_1 \cup U$. Now, extend $\mathcal{G}'_{|X_2 \cap (X_1 \cup U)}$ to a coherent sheaf $\mathcal{G}_2$ on X_2. Conclude by gluing $\mathcal{G}_1$ and $\mathcal{G}_2$ to a coherent sheaf on X.*

As an application, show that any quasi-coherent sheaf on a Noetherian scheme is a direct colimit of sub-coherent sheaves.

Exercise to hand in. *Trivialization of tangent bundle for cubics.* (Due Wednesday December 3, 12:00) Please write your solution in TEX.

(1) Let k be a field. Recall that $\Omega_{\mathbb{P}^n_k/k}$ is a locally free sheaf of rank n. Its restriction to $D_+(x_i)$ is free with basis elements $\{d(x_j/x_i)\}_{j\neq i}$:

$$\Omega_{\mathbb{P}^n_k/k}(D_+(x_i)) = \bigoplus_{i\neq j} k[x_j/x_i]_{i\neq j} d(x_j/x_i).$$

On $D_+(x_ix_j)$, where $i \neq j$, compute the base change matrix relating these two bases.

(2) Using the bases above and the base change matrix, prove that $\Omega_{\mathbb{P}^n_k/k}$ does not have any nonzero global sections.

(3) In the hand in exercise of week 6, we proved that the tangent sheeaf of a special cubic curve was trivial. In this exercise, we prove the same result for any cubic curve, but in a different way.

Let $C = \mathrm{Proj}(k[x, y, z]/f)$, where f is a irreducible homogeneous polynomial of degree three, and suppose that C is smooth over k. In other words, C is a smooth cubic curve in $\mathbb{P}^2_k$. Prove that $\Omega_C := \Omega_{C/k}$ is isomorphic to $\mathcal{O}_C$ following the steps below:

(a) Show that the ideal sheaf of $\mathbb{P}^2_k$ defining C is isomorphic to $\mathcal{O}_{\mathbb{P}^2_k}(-3)$.

(b) Let $\iota : C \to \mathbb{P}^2_k$ be the closed immersion. Show that the conormal sequence looks like

$$\iota^*\mathcal{O}_{\mathbb{P}^2_k}(-3) \to \iota^*\Omega_{\mathbb{P}^2_k} \to \Omega_C \to 0.$$

Prove that the left most arrow in the conormal sequence is in fact injective.

(c) Using determinants (Ex 1, week 10) and the Euler sequence (Ex 5, Week 9), deduce that $\Omega_C \cong \mathcal{O}_C$.

Note that using the same method one can compute Ω_X for any smooth hypersurface X in $\mathbb{P}^2_k$.