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AG II - Schemes and sheaves

Exercises – week 2

Exercise 1. *Sheaves of abelian groups.* Let X be a topological space. Let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphisms of sheaves of abelian groups.

(1) Let $\ker(\varphi)$ and $\operatorname{im}(\varphi)$ be respectively the kernel sheaf and the image sheaf.[1] Show that for every $x \in X$, one can define natural maps which are isomorphisms

$$\ker(\varphi)_x \to \ker(\varphi_x) \text{ and } \operatorname{im}(\varphi)_x \to \operatorname{im}(\varphi_x).$$

(2) Show that φ is an injective morphism of sheaves (*i.e.* injective on every open) if and only if for every $x \in X$ the morphism of abelian groups $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective

(3) Show that φ the following condition are equivalent.
   (a) For every $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective.
   (b) for every U open and $s \in \mathcal{G}(U)$, there exists an open cover $U = \bigcup U_i$ and sections $t_i \in \mathcal{F}(U_i)$ with $\varphi(t_i) = s_{U_i}$.

   In this case we say that the map is an *epimorphism of sheaves* or *a surjective map of sheaves*. This is is not to be mistaken with a surjective map of presheaves: the latter simply means that we have surjection on each open and this is strictly stronger than being an epimorphism of sheaves. See (6).

(4) Show that the natural map $\operatorname{im}(\varphi) \to \mathcal{G}$ is injective.

(5) Show that φ is an isomorphism if and only if it is an injective morphism of sheaves and a surjective morphism of sheaves.

(6) Let $f = X \to *$ be the unique morphism to the point. Show that $f_* = \Gamma(X, -) : \mathrm{Sh}_{\mathrm{Ab}}(X) \to \mathrm{Ab}$ is left-exact. Give an example to show that f_* is not right-exact in general.

Exercise 2. *Gluing sheaves.* Let X be a topological space and $\bigcup U_i = X$ an open cover of X. Let $(\mathcal{F}_i \in \mathrm{Sh}(U_i), \varphi_{ij})$ be a collection of sheaves on $\mathrm{Sh}(U_i)$ together with isomorphisms

$$\varphi_{ij} : \mathcal{F}_{i|U_{ij}} \xrightarrow{\sim} \mathcal{F}_{j|U_{ij}}$$

in $\mathrm{Sh}(U_{ij})$ satisfying for each i that $\mathrm{id} = \varphi_{ii}$ and for each i, j, k the following *cocycle condition* $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$.

Show that there exists a unique[2] sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ with maps $\psi_i : \mathcal{F}_{|U_i} \to \mathcal{F}_i$ with the following universal property: for all sheaves $\mathcal{G} \in \mathrm{Sh}(X)$ we have a

[1]The kernel sheaf is the kernel presheaf but the image sheaf is the *sheafification* of the image presheaf.

[2]up to isomorphism.

a bijection

$$\mathrm{Hom}(\mathcal{G},\mathcal{F}) \cong \left\{ (\mathcal{G}_{|U_i} \xrightarrow{f_i} \mathcal{F}_i) \in \prod_i \mathrm{Hom}(\mathcal{G}_{|U_i},\mathcal{F}_i) \mid \text{s.t. for all } i,j : \varphi_{ij} f_i = f_j \right\}$$

given by $f \mapsto \psi_i f_{|U_i}$.
Show furthermore that ψ_i are isomorphisms.

Remark. Can you see how the last exercise resembles the following statement: "$U \mapsto \mathrm{Sh}(U)$ is a sheaf"?

Exercise 3. *Inverse image.* Let $f\colon X \to Y$. Let $\mathcal{F} \in \mathrm{Sh}(Y)$. We define the presheaf on X

$$f^\sharp \mathcal{F}(U) = \varinjlim_{V \supset f(U)} \mathcal{F}(V).$$

(1) Show that if $f\colon * \to X$ is a point $x \in X$ then $f^\sharp \mathcal{F} = \mathcal{F}_x$.
(2) Show that if $y = f(x)$ then there is a natural isomorphism

$$(f^\sharp \mathcal{F})_x \to \mathcal{F}_y.$$

(3) Show that if f is an open immersion, then $f^\sharp$ is a sheaf.
(4) Find an example of map of topological spaces $f : X \to Y$ and a sheaf $\mathcal{F}$ on Y such that $f^\sharp \mathcal{F}$ is *not* a sheaf.
(5) Let $f^{-1}\mathcal{F}$ be the sheafification of $f^\sharp \mathcal{F}$. We call this sheaf the *inverse image* of $\mathcal{F}$. Show that the $f^{-1} \dashv f_*$[3] meaning that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{F}, f_*\mathcal{G}).$$

Exercise 4. *Localization* Let R be a ring. Let S be a multiplicative subset.
(1) Describe the points of $\mathrm{Spec}(S^{-1}R)$. If $\mathfrak{p} \in \mathrm{Spec}(R)$ show that $\mathrm{Spec}(R_\mathfrak{p})$ is the intersection of all opens containing $\mathfrak{p}$.
(2) Let M be an R-module and $I \supseteq R$ an ideal. Show that there is an isomorphism

$$S^{-1}(M/I) \cong (S^{-1}M)/(IS^{-1}M).$$

(3) Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and $I \leq R$ and ideal. When

$$(R/I)_\mathfrak{p} = 0 \quad ?$$

Can you interpret this geometrically?
(4) Let R be integral. Identify the image of the injective map $S^{-1}R \to \mathrm{Frac}(R)$.

Exercise 5. *Affine schemes are quasi-compact.* Let R be a ring. Show that $\mathrm{Spec}(R)$ is quasi-compact.[4] Deduce that the underlying topological space of any (affine) scheme has a basis of quasi-compact open subsets.

[3]We say that f^{-1} is *left adjoint to* f_*

[4]A topological space X is *quasi-compact* if every open cover of X can be refined to a finite cover.

Exercise 6. *Connected affine schemes.* We say that a ring R is *connected* if for all $a, b \in R$ if

$$a + b = 1 \text{ and } ab = 0$$

then exactly one of the two elements is non-zero.

(1) Show that it is equivalent to the fact there is exactly two idempotents (namely 0 and 1) in the ring R.
(2) Show that R is connected if and only if $\mathrm{Spec}(R)$ is connected.

Exercise 7. *Stalks, morphisms and cotangent spaces*

(1) Let R be an integral domain. Consider $\varphi\colon R[x,y] \to R[x,y]$ defined by $x \mapsto xy$ and $y \mapsto y$. Consider

$$f\colon \mathrm{Spec}(R[x,y]) \to \mathrm{Spec}(R[x,y])$$

the induced map on associated affine schemes.[5] Show that for all $\lambda \in R$ we have $f((x-\lambda, y)) = (x, y)$.
(2) Let now $R = k$ a field. Consider the induced map on local rings

$$k[x,y]_{(x,y)} \to k[x,y]_{(x-\lambda,y)}.$$

We write $\mathfrak{m}_{(0,0)} := \mathfrak{m}_{(x,y)}$ and $\mathfrak{m}_{(\lambda,0)} := \mathfrak{m}_{(x-\lambda,y)}$ for the maximal ideals of these local rings. Understand the induced k-linear map

$$\mathfrak{m}_{(0,0)}/\mathfrak{m}^2_{(0,0)} \to \mathfrak{m}_{(\lambda,0)}/\mathfrak{m}^2_{(\lambda,0)}.$$

This mean the following: find a k-basis of these vector spaces and describe the matrix of the map in term of your chosen basis.
Remark. We will later see that these vector spaces are the *cotangent spaces* at $(0,0)$ and $(\lambda, 0)$ respectively and that the map that you studied is *the precompostion by the differential of f at these points.*

Exercise to hand in. *An elliptic curve is not rational* (Due Wednesday October 1, 12:00) Please write your solution in TEX.
Let k be an algebraically closed and $2 \neq 0$ in k. Let A and B be finitely generated k-algbras which are domains.
We say that $A \to B$ *induces a birational map on* Spec if it injective and induces an isomorphism at fraction fields.

(1) In the above case, show that there is $a \in A$ such that the localization

$$A_a \to B_a$$

is an isomorphism. How do you intepret this geometrically?

We say that a k-finitely generated domain A is the algebra of functions of an affine *k-rational curve*[6] if there is a k-algebra isomorphism

$$\mathrm{Frac}(A) \cong k(t).$$

Background for question (2).

[5]Recall that the induced map on Spec is given by the *preimage* φ^{-1}
[6]The curve in question is $\mathrm{Spec}(A)$.

- We say that a domain A is normal if $A \subset \text{Frac}(A)$ is integrally closed. Moreover, a ring is normal if and only

$$A = \bigcap_{\substack{A \subset D \subset \text{Frac}(A) \\ D \text{ is a DVR}}} D$$

When A is of dimension 1, each D is of the form $A_{\mathfrak{m}}$ for a maximal ideal $\mathfrak{m} \subset A$.
- If $k \subset D \subset k(t)$ is a DVR, then $D = D_\lambda$ for a $\lambda \in k \cup \infty$ where

$$D_\lambda = \begin{cases} k[t]_{(t-\lambda)} & \lambda \neq \infty \\ k[\frac{1}{t}]_{(\frac{1}{t})} & \lambda = \infty \end{cases}$$

(2) Show the following lemma.

Lemma 1. Let A the algebra of an affine k-rational curve. Suppose that A is normal. Then A is an UFD.

Hint: Use that A is a finitely generated k-algebra to show that there is a non empty finite subset $Z \subset k \cup \infty$ such that $A \not\subset D_\lambda$. Then using that A is normal and the above remarks, conclude that

$$A = \bigcap_{\lambda \in k \cup \infty \setminus Z} D_\lambda$$

and recognize this ring as a localization of $k[\frac{1}{t-\mu}]$ for any $\mu \in Z$.[7]

Now, we consider

$$A = k[x, y]/(y^2 - (x^3 + x)).$$

This is the algebra of functions of an affine k-*ellpitic curve.*
Let

$$F = k(x)[y]/(y^2 - (x^3 + x))$$

the fraction field of A. You can assume that this is the fraction field of A without proof.

(3) Show that A is normal.
Hint: Let σ be the involution of F which fixes the field k, fixes x and sends y to $-y$. Denote by B the integral closure of A in F. First, show that if $b \in B$, then $\sigma(b) \in B$. Now using that any element in B is written as $b = ya(x) + b(x)$ for $a(x), b(x) \in k(x)$, show using that $\sigma(b) \in B$ that $b(x) \in B$. Deduce that $b(x) \in k[x]$. Then show that $a(x)$ is in $k[x]$ using that now we know that $ya(x)$ and $\sigma(ya(x)) = -ya(x)$ are in B, so $-y^2a(x)^2 = (x^3 + x)a(x)^2 \in B$ – deduce that this last element is also in $k[x]$. Then using that $k[x]$ is a UFD, deduce that $a(x) \in k[x]$. Conclude that $A = B$.

(4) Show that A is not an UFD. Therefore, A is not the algebra of functions of a k-rational curve.
Hint: in an UFD, an element is irreducible if and only the ideal generated by it is prime.

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[7]By convention if $\mu = \infty$ we say that $t = \frac{1}{t-\infty}$