Léo Navarro Chafloque AG II - Schemes and sheaves

Blow-ups.

1. Introduction

Blowing-up is a construction with many faces and many uses. We will try to expose some of these features in this introduction. In this introduction every scheme appearing is a scheme over an algebraically closed field.

1.1. **Replacing a closed subscheme with directions out of it.** Take a scheme X and a closed subscheme Z. The first description of the *blow-up of X at Z* is a scheme

$$b\colon \mathrm{Bl}_Z \to X$$

with a map to X which is an isomorphism on the complement of Z in the sense that

$$b\colon \mathrm{Bl}_Z \setminus b^{-1}(Z) \to X \setminus Z,$$

and the map

$$b^{-1}(Z) \to Z$$

is the "space of directions" out of Z. We wish to make more clear this last sentence in a first example.

Example 1.1. We take X to be $\mathbb{A}^2$ and $Z = V(x, y)$ the origin, that we denote by 0. A description of Bl_0 on points is the following closed subscheme[1] of $\mathbb{A}^2 \times \mathbb{P}^1$

$$\mathrm{Bl}_0 = \{(x, l) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid x \in l\}.$$

If $x \neq 0$, there is a unique linear line that goes through x, explaining why the projection is an isomorphism on $\mathbb{A}^2 \setminus 0$. However, the linear lines all goes through 0 so the preimage of 0 is $\mathbb{P}^1$. We see here that we replaced 0 by *directions* out of 0 which is here the set of lines out of 0.

1.2. **Resolving indeterminacies.** Blow-up can also be useful to extend maps that are not defined everywhere. Namely if $U \subset X$ is an open set and $U \to \mathbb{P}^n$ is a map, we want to find a way to "extend the map" outside of U" to a scheme $\widetilde{X} \to \mathbb{P}^n$. The relation of $\widetilde{X}$ with X should be given by a map $b\colon \widetilde{X} \to X$ such that $b^{-1}(U) \to U$ is an isomorphism.
For example if $U = \mathbb{A}^2 \setminus 0 \subset \mathbb{A}^2$ and

$$\mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$$

is the quotient map, we can try to "extend the map". Note that every point which is in the same line is sent to the same point in $\mathbb{P}^1$. Therefore if we want to extend this map, in $\widetilde{\mathbb{A}^2}$ there should be at least one new point per line, which should be the limit of the non-zero points in the line.
A way to extend a map is to look at the closure of the graph in $\mathbb{A}^2 \times \mathbb{P}^1$, and then look at the first projection. Note that with the above map we exactly get the blow-up of $\mathbb{A}^2$ at the origin, the example displayed in the last section.

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[1] This subscheme is also called the *canonical bundle on* $\mathbb{P}^1$

1.3. **Resolving singularities.** Blow-up can also serve to resolve singularities. We take the example of a curve which is singular at one point. If the curve is singular, this means that the tangent space at this point is of dimension strictly greater than 1, so that there are more than one tangent directions. Blowing-up this point will "separate the directions" in the sense that it will replace the point by as many points that they are "directions out of the point. This will reduce the dimension of the tangent space at these point, and with possibly many blow-ups we will find a regular curve and a map to the singular which an isomorphism on an open set. This is what we call a *resolution of singularities.* There is an example of such a blow-up at the end of this document.

Remark. Hironaka showed in 1964 that if k is a field of characteristic zero, any separated and finite type scheme over $\mathrm{Spec}(k)$ admits a resolution of singularities as a sequence of blow-ups. The problem of resolution of singularities over an arbitrary base field is one of the most important open problems in algebraic geometry.

1.4. **Turning an ideal principal.** The last point of view that we mention in this introduction about blowing-up can be understood as the *universal way of turning an ideal locally principal.* Namely if I is an ideal in R, then in Bl_I the pre-image of $V(I)$ will be a closed subscheme $E \subset \mathrm{Bl}_I$ that is locally principal in the following sense: for sufficiently small open affines $\mathrm{Spec}(A) \subset \mathrm{Bl}_I$, the closed subscheme $E \cap \mathrm{Spec}(A)$ is given by an ideal $J = (j)$ with j being a non-zero divisor (so that $(j) \cong A$ as modules).
Geometrically this can be interpreted as follows: blowing-up is the universal way of turning any closed subscheme into a *codimension 1 closed subscheme.* Recall that closed subschemes cut by one non-zero divisor are codimension 1 closed subschemes by Krull's height theorem.

2. Definitions and key properties

2.1. **Definitions.** We will now define blow-ups in the language of schemes. Let $X = \mathrm{Spec}(R)$ be an affine scheme and I be an ideal. We denote by $Z = V(I)$ the closed subscheme associated to I and $U = X \setminus Z$.

Definition 2.1 (Blow-up)**.** The *blow-up of X at Z* is

$$b\colon\ \mathrm{Proj}(\bigoplus_{n\geq 0} I^n) \to X$$

where $\bigoplus_{n\geq 0} I^n$ is the graded algebra with I^n placed in degree n with $I^0 = R$.

Remark. We can realize the blow-up algebra

$$\bigoplus_{n\geq 0} I^n$$

(also called the *Rees algebra of I*) by the sub-graded algebra of $R[T]$

$$\bigoplus_{n\geq 0} I^n t^n \subset R[T]$$

where the grading comes from the T-grading.

Example 2.2. We begin by a rather uninteresting example but realizing this will be useful for the first key property. If $I = R$, (so $Z = \emptyset$) then $\bigoplus_{n\geq 0} I^n = \bigoplus_{n\geq 0} R$. One sees that sending t to $1^{(1)} \in R$ in degree 1 gives a graded isomorphism

$$R[t] \to \bigoplus_{n\geq 0} R.$$

Therefore $\mathrm{Bl}_R = \mathrm{Proj}(R[t]) = \mathrm{Spec}(R)$.

Definition 2.3 (Exceptional divisor)**.** The exceptional divisor E is defined to be the fiber product of $(Z \to X \leftarrow \mathrm{Bl}_I)$. In other words this is $b^{-1}(Z)$ with it's natural scheme structure. It is therefore a closed subscheme of X. Note that by the compatibility of Proj with pullbacks this can be described as

$$\mathrm{Proj}(\bigoplus_{n\geq 0} I^n/I^{n+1}).$$

2.2. **Key properties.** We will show some properties and explain why some features of the introduction are indeed met by this construction.

(1) We show that b induces an isomorphism

$$b\colon b^{-1}(U) \to U.$$

To see this, let $f \in I$ so that $D(f) \subset U$. Note that $I_f = R_f$. Therefore by the compatibility of Proj and pullbacks we have as in the example above

$$b^{-1}(U) = \mathrm{Proj}(\bigoplus_{n\geq 0} R_f) = \mathrm{Proj}(R_f[t]) = \mathrm{Spec}(R_f).$$

As U is covered by such $D(f)$'s the above map is locally an isomorphism and therefore an isomorphism.

(2) The exceptional divisor can be interpreted in good cases as the spaces of "directions" outside Z. By this, we mean that each point of E should correspond to a directions in X which cut transversely to Z. Namely, I/I^2 is in good cases the *co-normal bundle.* We will see that therefore $E = \mathbb{P}(\mathcal{N}_{Z|X})$ the *projective bundle associated to the normal bundle.* We have already seen an important case where we know how to interpret I/I^2 as a space of *directions out of* Z. Namely the case when Z is a k-rational and X is a k-scheme for a field k. We have seen that $\mathfrak{m}_x/\mathfrak{m}_x^2$ is to be interpreted as the *co-tangent space* at the point.

2.2.1. *Blowing-up the origin in the affine space.* We now concentrate on describing the case $R = A[x_0, \ldots, x_n]$ and $I = (x_0, \ldots, x_n)$. We denote this blow-up by Bl_0.

(1) *Standard charts.* Let us now consider The goal is to understand more clearly the blow-up algebra

$$\bigoplus_{n\geq 0} I^n$$

in this case.

Note that we have the following surjective morphism of graded $A[x_0, \dots, x_n]$-algebras

$$A[x_0, \dots, x_n, Y_0, \dots, Y_n] \to \bigoplus_{n \geq 0} I^n$$

sending[2] Y_i to $x_i^{(1)}$. We claim that the kernel of this map is

$$(x_i Y_j - x_j Y_i).$$

See the appendix A for a detailed proof. The key is that $x_0, \dots, x_n$ form a regular sequence.

We then conclude that

$$\mathrm{Bl}_0 \cong V_+(x_i Y_j - x_j Y_i) \subset \mathbb{A}^{n+1}_A \times_A \mathbb{P}^n_A$$

(2) *Standard affine charts.* We furthermore investigate the last point and exhibit affine charts. We denote by $D_+(Y_k)$ the open of Bl_0 coming from the construction as a Proj. One finds that the equations defining the ideal become

$$(x_j \frac{Y_i}{Y_k} - x_i \frac{Y_j}{Y_k})$$

after localizing by Y_k. In particular we have for $j \neq k$ that $x_i = x_k \frac{Y_i}{Y_k}$. Note that when $i, j \neq k$ we have then that $x_j \frac{Y_i}{Y_k} = x_k \frac{Y_j}{Y_k} \frac{Y_i}{Y_k} = x_k \frac{Y_i}{Y_k} \frac{Y_j}{Y_k} = x_i \frac{Y_j}{Y_k}$. Therefore we can keep only the equations for $i \neq k$

$$x_i - x_k \frac{Y_i}{Y_k}.$$

Therefore functions on $D_+(Y_k)$ are given by

$$A[x_0, \dots, x_n, \frac{Y_0}{Y_k}, \dots, \frac{Y_n}{Y_k}]/(x_i - x_k \frac{Y_i}{Y_k})_{i \neq k}.$$

But sending $x_i \mapsto x_i$ and $\frac{Y_i}{Y_k} \mapsto \frac{x_i}{x_k}$ gives an isomorphism to

$$A[x_k, \frac{x_i}{x_k}]_{i=0,\dots,n}.$$

This last ring is a polynomial ring on A in $n+1$ variables so we conclude that

$$D_+(Y_k) \cong \mathbb{A}^{n+1}_A.$$

Note also that on $D_+(Y_k Y_{k'})$ we get that functions identifies to

$$A[x_k, x_{k'}, \frac{x_i}{x_k}, \frac{x_j}{x_{k'}}]_{i,j=0,\dots,n}$$

and that $D_+(Y_k Y_{k'}) \to D_+(Y_k)$ is induced by the obvious inclusion at the level of rings.

We conclude that

$$\mathrm{Bl}_0 = \bigcup_{k=0}^{n} \mathrm{Spec}\left(A[x_k, \frac{x_i}{x_k}]_{i=0,\dots,n}\right) \tag{1}$$

along the natural gluing maps displayed just above.

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[2] the notation $^{(1)}$ meaning "placed in degree 1"

(3) *Locally principal.* Note that the ideal $(x_0, \dots, x_n)$ turns into the principal ideal (x_k) under

$$A[x_0, \dots, x_n] \to A[x_k, \frac{x_i}{x_k}]_{i=0,\dots,n}$$

which is the algebraic counterpart of $b\colon D_+(Y_k) \to \mathbb{A}^{n+1}_A$.

(4) *Exceptional divisor.* Still in the above case, note that the exceptional divisor which is the fiber of $(x_0, \dots, x_n)$ is then given by $V_+(x_0, \dots, x_n, x_iY_j - x_jY_i) = V_+(x_0, \dots, x_n)$ so is (using the compatibility of Proj and base change)

$$\operatorname{Proj}(A[Y_0, \dots, Y_n]) = \mathbb{P}^n_A.$$

2.3. **Strict transforms.** Let $X = \operatorname{Spec}(R)$, $Z = V(I)$ and $C = V(J)$ another closed subscheme of X.

Definition 2.4. The *strict transform* St_J of J with respect to the blow-up $b\colon \mathrm{Bl}_I \to \operatorname{Spec}(R)$ is the blow-up of $C \cap Z = V(I + J/J)$ in $\operatorname{Spec}(R/J) = C$.

Remark. This is called the *strict transform* in opposition with the *total transform* which is defined to $b^{-1}(C)$. The strict transform is always a closed subscheme of the total transform. (See the remark below).

Remark. By definition the strict transform St_J is

$$\operatorname{Proj}(\bigoplus_{n\geq 0} (I + J)^n/J)$$

As the kernel of $I^n \to (I + J)^n/J$ is $I^n \cap J$ we see that we can realize the strict transform as the closed subscheme of Bl_I given by $V_+(\bigoplus_{n\geq 0} I^n \cap J)$. In opposition the *total transform* is $V_+(\bigoplus_{n\geq 0} I^nJ)$ using compatibility of Proj and base change.

Example 2.5. We give an example of strict transform which will also show how blow-ups can resolve singularities.

Let k be a field and consider $R = k[x_0, x_1]$, $I = (x_0, x_1)$ and $J = (x_1^2 - (x_0^3 + x_0^2))$ which is a singular plane curve, which is the called the *node*. We compute the strict transform St_J. We claim that $\mathrm{St}_J \cong \mathbb{A}^1_k$ (which is regular) and that the blow-up map may be described as $\mathbb{A}^1_k \to C \subset \mathbb{A}^2_k$

$$\lambda \mapsto (\lambda^2 - 1, \lambda^3 - \lambda).$$

We use the standard charts, meaning that we see $\mathrm{Bl}_I \subset \mathbb{A}^2 \times \mathbb{P}^1$. Recall that this inclusion is induced by the surjection

$$k[x_0, x_1, Y_0, Y_1] \to \bigoplus_n I^n$$

sending Y_i to x_i in degree 1. We claim that the preimage of the ideal

$$V_+\left(\bigoplus_n I^n \cap J\right)$$

by the above map is given by $(H, (x_1Y_0 - x_0Y_1))$ where

$$H = (x_1^2 - (x_0^3 + x_0^2), x_1Y_1 - (x_0^2Y_0 + x_0Y_0), Y_1^2 - (x_0Y_0^2 + Y_0^2))$$

Indeed, for degree zero, one and two, these elements are sent to the generator of J (we have $I^n \cap J = J$ for $n \leq 2$).
We now argue that these generators are enough. Note that being in I^n for a polynomial means that the monomials forming it are at least of degree n. Being in J means that the polynomial is of the form $f(x_0, x_1)(x_1^2 - (x_0^3 + x_0^2))$ for an $f(x_0, x_1) \in k[x_0, x_1]$. So we see that if such an element $f(x_0, x_1)(x_1^2 - (x_0^3 + x_0^2))$ is in $I^n \cap J$, then $f(x_0, x_1) \in I^{n-2}$ counting the degrees of the monomials because $(x_1^2 - (x_0^3 + x_0^2)) \in I^2 \setminus I^3$. Therefore for $n \geq 3$ using the degree 2 generator and elements of I in degree 1, we can attain every element of $I^n \cap J$.
Therefore the strict transform is

$$\mathrm{Proj}(k[x_0, x_1, Y_0, Y_1]/(H, x_0Y_1 - x_1Y_0)).$$

Denote by B the grading ring we are taking Proj of. Note that $V_+(H, x_0Y_1 - x_1Y_0, Y_0) = \emptyset$, so that $V_+(H, x_0Y_1 - x_1Y_0, Y_0) \subset D_+(Y_0)$ implying that

$$\mathrm{Proj}(k[x_0, x_1, Y_0, Y_1]/(H, x_0Y_1 - x_1Y_0)) = \mathrm{Spec}(B_{(Y_0)}).$$

But, if we write $\frac{Y_1}{Y_0}$ by y we get

$$\begin{aligned} B_{(Y_0)} &= k[x_0, x_1, y]/(x_1^2 - (x_0^3 + x_0^2), x_1y - (x_0^2 + x_0), y^2 - (x_0 + 1), (x_0y - x_1)) \\ &\cong k[x_0, y]/(y^2 - (x_0 + 1)) \cong k[y]. \end{aligned}$$

Indeed using $x_0y = x_1$ the equation $x_1^2 - (x_0^3 + x_0^2)$ turn into $x_0^2(y^2 - (x_0 + 1))$ and $x_1y - (x_0^2 + x_0)$ turn into $x_0(y^2 - (x_0 + 1))$ which are both subsumed by the equation coming from degree 2.
Therefore we see that the strict transform is isomorphic to $\mathbb{A}^1$. By using that under this isomorphisms $x_0 \mapsto y^2 - 1$ and $x_1 \mapsto x_0y = y^3 - y$ the claim about the form of the map follows.
If $k = \mathbb{C}$ something really intuitive happens on real points. Namely note that solutions to J are given by Figure 1. Then, the map $\mathbb{A}_k^1 \to C$ is understood to be a parameterization of the curve where both 1 and -1 are sent to the origin. We can therefore picture this as in Figure 2.

We now give a generalization of the computation above in Example 2.5, which actually boils down to the same reasoning.

Proposition 2.6 (Computing strict transforms)**.** *Let A be a ring. Let $f \in A[x_0, \ldots, x_n]$ be a non-zero polynomial. We define*

$$d := \min\{\textit{degree of a monomial appearing in } f\} \in \mathbb{N}.$$

Let $I = (x_0, \ldots, x_n) \subset A[x_0, \ldots, x_n]$ and $J = (f)$. Then the srict transform St_J *with respect to the blow-up $\mathrm{Bl}_I \to \mathbb{A}_A^n$ is given by the gluing*

$$\bigcup_{k=0}^{n} \mathrm{Spec}\left(\frac{A[x_k, \frac{x_i}{x_k}]_{i=0,\ldots,n}}{x_k^{-d}f}\right).$$

In other words,

$$x_k^{-d}f \in A\left[x_k, \frac{x_i}{x_k}\right]_{i=0,\ldots,n}$$

is the equation defining the strict transform in the standard chart indexed by k from Equation (1)*.*

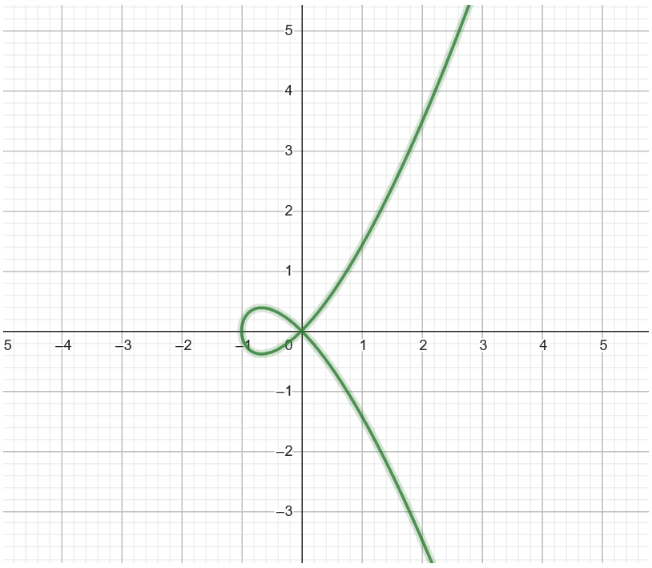

FIGURE 1. The real points of the cusp.

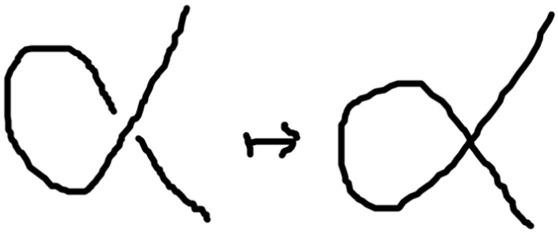

FIGURE 2. An horrible paint drawing of what is meant to happen. You can check that on real points this is a depiction of $\lambda \mapsto (\lambda^2 - 1, \lambda^3 - \lambda)$.

Proof of Proposition 2.6. By Definition 2.4, St_J is given by $V_+(\bigoplus_{n\geq 0} I^n \cap J)$ in the Proj of the graded ring

$$\bigoplus_{n\geq 0} I^n$$

defining Bl_I. We know from the discussion in Section 2.2.1 that we have an isomorphism of graded rings

$$A[x_0, \ldots, x_n, Y_0, \ldots, Y_n]/(x_iY_j - x_jY_i) \to \bigoplus_{n\geq 0} I^n \tag{2}$$

and we therefore want to understand the pre-image of the ideal $\bigoplus_{n\geq 0} I^n \cap J$. It's important to have the following in mind for the rest of the proof: *a polynomial $g \in A[x_0, \ldots, x_n]$ is in I^n if and only if the degree of every monomial appearing in it is at least n.*

- Note that for $n \leq d$ we have

$$I^n \cap J = J.$$

- Note also the following. If $n > d$ and $g \in I^n \cap J$ then $g = g'f$ for some $g' \in A[x_0, \dots, x_n]$. By hypothesis, every monomial of g is of degree at least n. Now, if m is a monomial appearing in f of degree d, we have that $g'm$ is constituted of monomials of degree at least n, implying that monomials appearing in g' are of degree at least $n - d \geq 1$. In other words we conclude that whenever $g \in I^n \cap J$, for a g' such that $g = g'f$ we have $g' \in I^{n-d}$.

Therefore for $n \leq d$ take an element of degree n in $A[x_0, \dots, x_n, Y_0, \dots, Y_n]$ sent to f in degree n by the map in Equation (2), and denote it $f_{(n)}$, with $f_{(0)} = f$. We claim that the pre-image of $\bigoplus_{n \geq 0} I^n \cap J$ is given by

$$(f_{(n)})_{n \leq d}.$$

Indeed:

- For $n \leq d$ the generators of this ideal are precisely sent to generator of $J = I^n \cap J$.
- For $n > d$, we explained above that any element $g \in I^n \cap J$ was written has $g'f$ for some $g' \in I^{n-d} \subset A[x_0, \dots, x_n]$. To attain $g'f$ it suffices to attain mf where $m \in I^{n-d}$ is a monomial appearing in g'. Say $m_{(n-d)}$ is a pre-image of m in degree $n - d$ in $A[x_0, \dots, x_n, Y_0, \dots, Y_n]$. Therefore we see that we can attain mf in degree n from $m_{(n-d)} f_{(d)}$, concluding.

Now it remains to inspect what happens at homogeneous localizations to understand the statement of the proposition. For k fixed, the ideal is therefore given by

$$(Y_k^{-n} f_{(n)})_{n \leq d}.$$

But note that as (because of the relation $(x_i Y_j - x_j Y_i)$)

$$x_k^n f_{(n)} = Y_k^n f,$$

we deduce that the above ideal is generated by $x_k^{-d} f$, as desired.

□

Appendix A. Equations of standard charts

Definition A.1 (Regular sequence)**.** Let R be a ring. A finite sequence of elements $f_1, \dots, f_n$ is said to be a *regular sequence* if f_i is a non-zero divisor in $R/(f_1, \dots, f_{i-1})$ and $R/(f_1, \dots, f_n)$ is non-zero.

Example A.2. Let A be a ring and $R = A[x_0, \dots, x_n]$. Then $x_0, \dots, x_k$ for $0 \leq k \leq n$ is a regular sequence. If $f_1, \dots, f_k$ is a regular sequence in a general ring, then for integers $e_i > 0$ the sequence $f_1^{e_1}, \dots, f_k^{e_k}$ is also a regular sequence.

We show the following.

Proposition A.3. *Let R be a ring and $I = (f_1, \dots, f_n)$ where $f_1, \dots, f_n$ form a regular sequence. Then the kernel of the surjection sending Y_i to $f_i^{(1)}$*

$$R[Y_1, \dots, Y_n] \to \bigoplus_{n \geq 0} I^n$$

is given by the ideal $J = (f_i Y_j - f_j Y_j)$.

Proof. We show this using two steps which both really heavily on the regular sequence hypothesis.
First, we show that the kernel of

$$R^n \to I$$

sending $e_i \mapsto f_i$ is generated by the vectors $e_i f_j - e_j f_i$. We proceed by induction on n the length of the regular sequence. For $n = 0, 1$ the claim is obvious. To proceed inductively we define the chain complex K_n

$$R^{\binom{n}{2}} \to R^n \to R$$

with R placed in degree zero and differentials being respectively given by $e_{i,j} \mapsto f_j e_i - f_j e_i$ and $e_i \mapsto f_i$. Note that $H_0(K_n)$ of this complex is $R/(f_1, \dots, f_n)$. and that the claim amounts to this complex being exact in the middle, meaning that $H_1(K_n) = 0$. Note that we also have the following exact sequence of complexes

$$\begin{tikzcd}
R^{\binom{n}{2}} \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d] & R \arrow[d, shift left] \\
R^{\binom{n+1}{2}} \arrow[r] \arrow[d] & R^{n+1} \arrow[r, shift left] \arrow[r, shift right, red] \arrow[d, shift right] & R \arrow[u, shift left, red] \arrow[d] \\
R^n \arrow[r] & R \arrow[u, shift right, red] \arrow[r] & 0
\end{tikzcd}$$

Where the left vertical arrows are given by $e_{i,j} \mapsto e_{i,j}$ and $e_{i,j} \mapsto \delta_{n+1,j} e_i$. The middle vertical arrows are $e_i \mapsto e_i$ and $e_i \mapsto \delta_{i,n+1}$. The first right vertical arrow is the identity.
By induction $H_1(K_n) = 0$, and we want to show that $H_1(K_{n+1}) = 0$. The long exact sequence in homology gives

$$0 = H_1(K_n) \to H_1(K_{n+1}) \to R/(f_1, \dots, f_n) \xrightarrow{\delta} R/(f_1, \dots, f_n),$$

where δ is the connecting morphism. The connecting morphism is computed by following the red arrows on the diagram above. It is therefore given by

$\delta = \cdot f_{n+1}$ the multiplication by f_{n+1}. As $f_1, \dots, f_{n+1}$ is a regular sequence δ is injective and therefore $H_1(K_{n+1}) = 0$.

Remark. These complexes are not taken out of nowhere. They are simplifications of *Koszul complexes*, a well known notion.

We have now understood the degree 1 elements of the kernel of the surjection sending Y_i to $f_i^{(1)}$

$$R[Y_1, \dots, Y_n] \to \bigoplus_{n \geq 0} I^n.$$

It now suffices to show that this kernel is generated by degree 1 elements. Just for the rest of this proof, we call a polynomial $F \in R[Y_1, \dots, Y_n]$ to be of *weight* i if i is the minimal integer such that $F \in (Y_1, \dots, Y_i)$ but $f(Y_1, \dots, Y_n) \notin (Y_1, \dots, Y_{i-1})$. A weight 0 polynomial is defined to be 0. This proposition will be shown as a special case (but the general case will be needed in the proof by induction) of the following.

Claim. Let $F \in R[Y_1, \dots, Y_n]$ be an homogeneous polynomial of degree m with

$$F(f_1, \dots, f_n) \in (f_1, \dots, f_k).$$

Then there exists an homogeneous polynomial G of degree m and weight at most k such that $F - G \in (f_i Y_j - f_j Y_j)$. In particular if $F(f_1, \dots, f_n) = 0$, then $F \in J = (f_i Y_j - f_j Y_j)$, showing the proposition.

We prove the claim by induction on the degree m of the polynomial. Let F be a polynomial of degree 1. Then

$$F(f_1, \dots, f_n) = \sum_{i=1}^{k} a_i f_i.$$

Therefore the weight k and degree 1 polynomial $G = \sum_{i=1}^{k} a_i Y_i$ satisfies the claim: indeed $F - G$ is an homogeneous polynomial of degree 1 with $(F - G)(f_1, \dots, f_n) = 0$. Therefore using the first part of the proof above, we see that $F - G \in J$.

Now if F is a polynomial of degree m, we show the claim by induction on the weight l of F. If $l \leq k$, set $F = G$. Otherwise write $F = Y_l F_1 + F_2$ with F_1 homogeneous of degree $m - 1$ and F_2 of weight at most $l - 1$. Recall that by hypothesis

$$f_l F_1(f_1, \dots, f_n) + F_2(f_1, \dots, f_n) \in (f_1, \dots, f_k) \subset (f_1, \dots, f_{l-1})$$

and $F_2(f_1, \dots, f_n) \in (f_1, \dots, f_{l-1})$ because F_2 is of weight at most $l - 1$ by construction. Modding out by $f_1, \dots, f_{l-1}$ and using that $f_1 \dots, f_l$ is a regular sequence we get that $F_1(f_1, \dots, f_n) \in (f_1, \dots, f_{l-1})$. We apply induction on the degree to get a polynomial G_1 of weight at most $l - 1$ such that $F_1 - G_1 \in J$. Now set $G' = Y_l G_1 + F_2$. This is a polynomial of weight at most $l - 1$ because G_1 and F_2 are. Note also that $F - G' = Y_l(F_1 - G_1) \in J$. Note also that $G'(f_1, \dots, f_n) = F(f_1, \dots, f_n) \in (f_1, \dots, f_k)$. By induction on the weight there is a polynomial G of weight at most k such that $G' - G \in J$. But now $F - G = (F - G') + (G' - G) \in J$, concluding the proof. $\square$