Cohomology Rings

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Content

Introduction

This is a Master's course taught for the first time at EPFL in the Fall semester 2023, and a second time in 2025. The following sequence of courses taught at the Bachelor's level constitute the natural background for the ideas we study here. In "Topology I, Metric and Topological Spaces" the notion of topological space is introduced as a generalization of that of a metric space, the fundamental group is defined and computed for the circle. In the sequel course "Topology II, The Fundamental Group" the whole semester is spent around this important homotopy invariant and two different aspects are highlighted. The Seifert-van Kampen Theorem allows us to compute fundamental groups of spaces constructed by assembling elementary pieces, in particular in the case of quotient spaces, and the theory of coverings gives a more geometrical meaning to the fundamental group by identifying its elements as deck transformations. Finally, the course "Topology III, Homology" makes use of homological algebra and introduces homology groups as a new homotopy invariant, both in the form of singular homology and that of cellular homology, a version better suited for computations when the spaces one works with are CW-complexes.

In this course we focus on a dual version of homology groups, namely cohomology groups. They have been introduced quickly in the "Topology III" course, and we will come back to them in this course. The main objective is to enhance the sequence of cohomology groups $H^n(X; \mathbb{Z})$ with a *graded ring structure*. As is often the case, more structure means stronger invariants. The product structure on singular or cellular cohomology also appears in the purely algebraic setting of group cohomology. The objective is thus to explain how these Ext groups can be endowed with a product and to compare both approaches by relating the category of groups with that of spaces through the so-called classifying space.

Several excellent textbooks serve as inspiration for this course. We do not claim any originality and rely often on the approach and technical tools presented in books by Rotman, Jeanneret and Lines, Brown, Switzer, tom Dieck, May, Adem and Milgram, etc. A short bibliography is available at the end of these notes, more precise references appear in the text as we go along.

CHAPTER 1

Group homology

This first chapter is devoted to groups and how one associates homological invariants to them. We follow Brown's book [**7**] for large parts.

1. Chain and cochain complexes

A non-negatively graded chain complex $C_\bullet$ in $\mathrm{Ch}(\mathbb{Z})$ is equipped with differentials that lower the degree by one:

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

It is often convenient to add artificially a trivial $\mathbb{Z}$-module $C_{-1} = 0$ and declare d_0 is the zero morphism. One requires that $d_n \circ d_{n+1} = 0$ for any $n \geq 0$ and defines then homology groups $H_n(C_\bullet) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$.

One important tool in homological algebra is given by long exact sequences in homology.

Proposition 1.1. *A short exact sequence* $0 \to C'_\bullet \to C_\bullet \to C''_\bullet \to 0$ *of chain complexes induces a long exact sequence*

$$\cdots \to H_{n+1}(C''_\bullet) \xrightarrow{\partial} H_n(C'_\bullet) \to H_n(C_\bullet) \to H_n(C''_\bullet) \xrightarrow{\partial} \dots$$

The connecting homomorphism ∂ is constructed by using the famous Snake Lemma. We will not (re)do the proof here, but we will use the explicit construction when needed.

In the category of chain complexes there are two important classes of morphisms. First, *homotopy equivalences* are chain maps admitting an inverse up to homotopy. They are characterized by the fact that their mapping cone is contractible. Second, weak equivalences or *quasi-isomorphisms* are chain maps inducing isomorphisms on all homology groups. The mapping cone is then acyclic, a weaker condition than contractibility.

Cohomology is defined from a cochain complex where differentials raise the degree by one:

$$0 \xrightarrow{d^0} C^0 \xrightarrow{d^1} C^1 \xrightarrow{d^2} C^2 \to \cdots \to C^{n-1} \xrightarrow{d^n} C^n \xrightarrow{d^{n+1}} C^{n+1} \to \dots$$

Here $H^n(C^\bullet) = \operatorname{Ker} d^{n+1} / \operatorname{Im} d^n$. Sometimes we write d^n_C to emphasize the fact that the differential belongs to the cochain complex $C^\bullet$.

Example 1.2. Any chain complex $C_\bullet$ dualizes to form a cochain complex of homomorphisms $\operatorname{Hom}(C_\bullet, \mathbb{Z})$ where C^n is the $\mathbb{Z}$-linear dual $\operatorname{Hom}(C_n, \mathbb{Z})$ and the differential $d^n = \operatorname{Hom}(d_n, \mathbb{Z})$.

Remark 1.3. Just like chain complexes, cochain complexes are the objects of a category whose morphisms $f^\bullet \colon C^\bullet \to D^\bullet$ are sequences of homomorphisms (for all n), $f^n \colon C^n \to D^n$ that are compatible with the differentials, i.e. $d_D^{n+1} \circ f^n = f^{n+1} \circ d_C^{n+1}$. In other words morphisms of cochain complexes are commutative ladders

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C^{n-1} & \xrightarrow{d_C^n} & C^n & \xrightarrow{d_C^{n+1}} & C^{n+1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \dots & \longrightarrow & D^{n-1} & \xrightarrow{d_D^n} & D^n & \xrightarrow{d_D^{n+1}} & D^{n+1} & \longrightarrow & \dots \end{array}$$

Any such morphism induces a homomorphism $H^n(f) \colon H^n(C^\bullet) \to H^n(D^\bullet)$. Cohomology groups form in fact functors from the category of cochain complexes to that of $\mathbb{Z}$-modules.

2. Groups rings

In this section we associate to each (discrete) group G a ring $\mathbb{Z}G$, called the group ring. Modules over $\mathbb{Z}G$ are abelian groups equipped with a G-action and we will be interested in resolutions of such modules by projective or free $\mathbb{Z}G$-modules in order to compute certain derived functors.

Definition 2.1. Let G be a group. The group ring $\mathbb{Z}G$ is the free $\mathbb{Z}$-module generated by the elements of G and the multiplication is determined by its effect on basis elements: $g \cdot h = gh$ for all $g, h \in G$.

One should maybe be more precise with the way the group multiplication determines the multiplicative structure of the group ring. Any element of $\mathbb{Z}G$ can be written as a finite sum $\sum n_g g$ with $n_g \in \mathbb{Z}$. Then a product of two elements is computed by $(\sum n_g g)(\sum m_h h) = \sum n_g m_h gh$.

Remark 2.2. This construction is not just some artificial invention allowing us to build new and strange rings out of groups. The underlying multiplicative group $R^\times$ of invertible elements (or units) in a ring R is a functor from the category $\mathsf{Ring} \to \mathsf{Group}$. The group ring construction provides a left adjoint, so that we have a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Z}G, R) \cong \mathrm{Hom}_{\mathsf{Group}}(G, R^\times)$$

Indeed the subset of basis elements $G \subset \mathbb{Z}G$ is contained in the units $(\mathbb{Z}G)^\times$ since every group element has an inverse, and a group homomorphism $f\colon G \to R^\times$ extends first to elements of the form ng by sending them to $n \cdot f(g) \in R$, and second to arbitrary elements $\sum n_g g$ by linearity, using the sum in R. This formula is compatible with the operations in the group ring by definition.

Interestingly, the long standing unit conjecture for group rings has been disproved recently, by Gardam in 2021, [**13**]. It stated that the units in the group ring KG over a filed K are all of the form λg when G is torsion free (here $\lambda \in K$ and $g \in G$). The Promislow group $\langle a, b \mid b^{-1}a^2b = a^2, a^{-1}b^2a = b^2 \rangle$ provides a counter example over the field $\mathbb{F}_2$!

Let us look at group rings of cyclic groups.

Example 2.3. To distinguish the operation in a group from the addition in the group ring, we will write the group multiplicatively, even when it is abelian. Let C by the cyclic group $\langle t \rangle \cong (\mathbb{Z}, +)$. The group ring $\mathbb{Z}C$ has t^k as a basis for its underlying $\mathbb{Z}$-module, for all $k \in \mathbb{Z}$. The product $t^k \cdot t^m$ is t^{k+m} so that $\mathbb{Z}C$ is isomorphic to the ring of *Laurent polynomials* $\mathbb{Z}[t, t^{-1}]$ whose elements are finite sums $a_{-m}t^{-m} + \cdots + a_{-1}t^{-1} + a_0 + a_1 t + \cdots + a_n t^n$, with $m, n \in \mathbb{N}$.

Example 2.4. Let C_2 be the cyclic group of order 2, generated by an element t such that $t^2 = 1$. As a $\mathbb{Z}$-module $\mathbb{Z}C_2$ is a direct sum of two copies of the integers $\mathbb{Z} \oplus \mathbb{Z}t$, where we abusively write n for $n \cdot 1$. Since $t^2 = 1$ and as the product is precisely defined as for polynomials, there is an isomorphism $\mathbb{Z}C_2 \cong \mathbb{Z}[t]/(t^2 - 1)$.

More generally $\mathbb{Z}C_n \cong \mathbb{Z}[t]/(t^n - 1)$ for any integer $n \geq 2$.

Remark 2.5. Group rings have appeared implicitly (one says) in early work of Cayley (1821-1895), [**9**]. This predates by 80 years Noether's work on rings and modules, but it is Cayley who first formalized the abstract notion of a group (not only as a group of permutations or symmetries). In fact, Cayley's Theorem has been named in his honour (that every group is a subgroup of a permutation group).

He worked for 14 years as a lawyer before getting a professorship in Cambridge. The Cayley-Hamilton Theorem we learn about in linear algebra has been verified by Cayley for matrices of dimension 2×2 and 3×3, the first general proof is due to Frobenius in 1878.

3. Modules over groups rings

Left $\mathbb{Z}G$-modules are also called G-modules for short and form a category $\mathrm{G}-\mathsf{Mod}$.

Lemma 3.1. *A G-module M is an abelian group equipped with a left G-action.*

Proof. An R-module is an abelian group together with a ring homomorphism $R \to \mathrm{End}(M)$ from R to the ring of endomorphisms of M. When $R = \mathbb{Z}G$ such a homomorphism corresponds by adjunction, see Remark 2.2, to a group homomorphism $G \to \mathrm{End}(M)^\times = \mathrm{Aut}(M)$. This is exactly a group action on M (on the left). □

Example 3.2. The $\mathbb{Z}$-module $\mathbb{Z}$ itself can always be equipped with the trivial G-action, thus yielding the *trivial module* $\mathbb{Z}$.

When $G = C_n$ is a cyclic group and n is even, $\mathbb{Z}$ can also be given another action, namely through the sign $t \cdot 1 = -1$. It corresponds to the only non constant group homomorphism $C_n \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$. We will write $\mathbb{Z}^\sigma$ for this C_n-module.

More generally any group representation $G \to \mathrm{Aut}(M)$ yields a permutation G-module through the above mentioned adjunction.

There is an important homomorphism of G-modules which we will use to start constructing resolutions in the next section.

Definition 3.3. The assignment $1_G \mapsto 1$ defines a unique G-module homomorphism $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$, called the *augmentation*. Its kernel IG is the *augmentation ideal*.

By definition the augmentation ideal IG contains all elements of the form $g-1$. They form a basis of the underlying free $\mathbb{Z}$-module, since $\sum n_g g$ belongs to IG if and only if $\sum n_g = 0$. One can indeed write

$$\sum n_g g = \sum n_g g - \sum n_g = \sum n_g (g-1)$$

4. Group homology and resolutions

The homology groups of a group G can be defined swiftly as the left derived functors of the tensor product over the group ring. We will be more precise and recall how one computes such derived functors by using appropriate resolutions. Since the tensor product is a functor of two variables there are more general versions of group homology than those we introduce in this first chapter. But let us start slowly.

Let G be a group, $\mathbb{Z}G$ its group ring, and let us view $\mathbb{Z}$ as a trivial *right* G-module. The tensor product we consider is the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} -\colon \mathsf{G{-}Mod} \to \mathsf{Ab}$.

Definition 4.1. Let M be a G-module. The group of *coinvariants* of M is the largest quotient M_G of M on which G acts trivially.

Explicitly, to form the coinvariants we identify the orbit Gm to a single element for any $m \in M$. More precisely M_G is the quotient of the $\mathbb{Z}G$-module M by the submodule generated by all elements of the form $gm - m$, for $g \in G$ and $m \in M$.

We identify both constructions, tensor product and coinvariants.

Lemma 4.2. *Let M be a G-module. The $\mathbb{Z}$-module $\mathbb{Z} \otimes_{\mathbb{Z}G} M$ is isomorphic to the group of coinvariants M_G of M.*

Proof. We notice that the augmentation $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$ induces a homomorphism $M \cong \mathbb{Z}G \otimes_{\mathbb{Z}G} M \to \mathbb{Z} \otimes_{\mathbb{Z}G} M$. It sends $gm - m$ to zero so that we get a comparison morphism $M_G \to \mathbb{Z} \otimes_{\mathbb{Z}G} M$. Instead of showing directly that this is an isomorphism we construct an inverse by invoking the universal property of the tensor product.

We define a map $\mathbb{Z} \times M \to M_G$ by sending the pair (k, m) to $k\overline{m}$, where $\overline{m}$ denotes the class of m in the quotient M_G. This map is clearly bilinear and moreover the two pairs $(k \times g, m) = (k, m)$ and (k, gm) have the same image. It induces therefore a morphism $\mathbb{Z} \otimes_{\mathbb{Z}G} M \to M_G$ which is the inverse of the previously defined morphism going the other way around. □

Remark 4.3. During his years as a lawyer for Lincoln's Inn in London, Cayley kept working on mathematics problems. He would often meet his friend Sylvester, at the time an actuary, living in London as a bachelor as well. They would take walks and talk about invariants and coinvariants. During these 14 years in London, Cayley wrote over 200 maths papers!

The tensor product is right exact, we are interested in its left derived functors.

Definition 4.4. Let G be a group. The *homology groups* $H_n(G;\mathbb{Z})$ of G are the left derived functors $\mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ of the coinvariants $\mathbb{Z}_G$.

Homological algebra comes in to give us a recipe to compute these homology groups. We choose a projective resolution $P_\bullet$ of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, tensor it over the group ring with $\mathbb{Z}$ to take coinvariants, and finally compute the homology groups of the resulting chain complex $(P_\bullet)_G$. Two projective resolutions are not isomorphic in general, but they are homotopic as chain complexes and so are their coinvariants. This implies that the above described procedure yields the same output independently of the choice of resolution.

We will primarily work with free resolutions, which always exist since any module M admits a free cover $\oplus\mathbb{Z}G \twoheadrightarrow M$. Of course if one is lucky enough to find a small resolution, computations are made easier.

Example 4.5. We have identified the group ring $\mathbb{Z}C_2$ with $\mathbb{Z}[t]/(t^2-1)$ in Example 2.4. We start constructing a free resolution $F_\bullet$ of the trivial module $\mathbb{Z}$ with the augmentation $\varepsilon\colon \mathbb{Z}[t]/(t^2-1) \to \mathbb{Z}$, see Definition 3.3. Its kernel is a free $\mathbb{Z}$-module of rank one, generated by $t-1$. We continue therefore with the map $t-1\colon \mathbb{Z}[t]/(t^2-1) \to \mathbb{Z}[t]/(t^2-1)$ that sends 1 to $t-1$ and the class of any polynomial $f(t)$ to that of $(t-1)\cdot f(t)$. This homomorphism of $\mathbb{Z}[t]/(t^2-1)$-modules surjects onto the augmentation ideal $(t-1)$ and has kernel generated by $(t+1)$. We obtain in this way a *periodic* resolution $F_\bullet$:

$$\dots \xrightarrow{t+1} \mathbb{Z}[t]/(t^2-1) \xrightarrow{t-1} \mathbb{Z}[t]/(t^2-1) \xrightarrow{t+1} \mathbb{Z}[t]/(t^2-1) \xrightarrow{t-1} \mathbb{Z}[t]/(t^2-1) \xrightarrow{\varepsilon} \mathbb{Z}$$

where $F_n = \mathbb{Z}[t]/(t^2-1)$ for all $n \geq 0$. We forget now the trivial module and keep the resolution, of which we take coinvariants, obtaining a copy of $\mathbb{Z}$ in each degree as we identify 1 and $t \cdot 1 = t$. On coinvariants $t-1$ induces the zero map, and $t+1$ induces multiplication by 2. We must therefore compute the homology of the complex

$$\dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Therefore $H_0(C_2;\mathbb{Z}) \cong \mathbb{Z}$, $H_{2n}(C_2;\mathbb{Z}) = 0$, and $H_{2n-1}(C_2;\mathbb{Z}) \cong \mathbb{Z}/2$ for all integers $n \geq 1$.

5. Free resolutions from topology

Suppose you know a contractible CW-complex X equipped with an action of G permuting freely the cells in each dimension. The cellular chain complex of this space is then a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Indeed, the augmented cellular chain complex $C_\bullet^{cell}(X) \xrightarrow{\varepsilon} \mathbb{Z}$ is acyclic (exact) because X is assumed to be contractible.

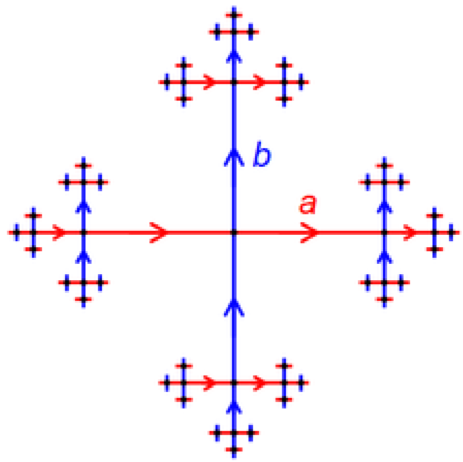
We know already such spaces from the theory of coverings in topology.

Example 5.1. The “infinite dimensional sphere” S^∞ is the union of all n-dimensional spheres S^n, each one seen as the equator in the next one S^{n+1}. There is a convenient cell decomposition with exactly two cells in each dimension, starting with S^0 (a discrete space of two points), and we add two hemispheres at each step. The antipodal map provides an action of the cyclic group C_2 which permutes freely the cells.

The augmented cellular chain complex $C_\bullet^{cell}(S^\infty;\mathbb{Z}) \xrightarrow{\varepsilon} \mathbb{Z}$ is isomorphic to the previously constructed free resolution from Example 4.5. Let us observe here that this infinite sphere is the universal cover of the infinite real projective space $\mathbb{R}P^\infty$. In fact this space is the quotient S^∞/C_2, so what we are looking at when we compute $H_*(C_2;\mathbb{Z})$ is the cellular homology of $\mathbb{R}P^\infty$.

Example 5.2. The real line $\mathbb{R}$ is equipped with a free action of $\mathbb{Z}$ by integral translations. We choose $\mathbb{Z} \subset \mathbb{R}$ as 0-cells, and segments $[n, n+1]$ as 1-cells for all integers $n \in \mathbb{Z}$. This yields a short resolutions $0 \to \mathbb{Z}C \to \mathbb{Z}C \to 0$ where the map sends the generator corresponding to the n-th 1-cell $[n, n+1]$ to $n+1$ minus n. In algebraic terms this corresponds to the map $t-1$ on Laurent polynomials $\mathbb{Z}[t, t^{-1}]$. Taking coinvariants yields the zero morphism on $\mathbb{Z}$, so that $H_0(C;\mathbb{Z}) \cong H_1(C;\mathbb{Z}) \cong \mathbb{Z}$ and all other homology groups are zero. Observe again that we have just computed the homology groups of the circle $S^1 = \mathbb{R}/\mathbb{Z}$, of which $\mathbb{R}$ is the universal cover.

Example 5.3. We use this time the universal cover X of a wedge of two circles, in the form of a Cayley graph of the free group $F(a, b)$ on two generators a and b.



This 1-dimensional CW-complex admits a free action of $F(a, b)$ on its 0-cells, which are in bijection with $F(a, b)$, so $C_0^{cell}(X; \mathbb{Z})$ is a copy of the group ring. There are two copies of the group ring in degree one, corresponding to the red and blue 1-cells on the above picture taken from Wikipedia (free group). Therefore $H_0(F(a, b); \mathbb{Z}) \cong \mathbb{Z}$, $H_1(F(a, b); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and all higher homology groups vanish.

What we have computed here is the homology of a wedge of two circles.

Remark 5.4. Cayley was finally elected as professor in Cambridge at the age of 42. He gave up a high salary for the joy of academic life. He decided to marry after so many years in London. All together his collected works contain 967 papers. Apparently his lectures were usually about his latest work.

6. The bar resolution

One can prove in homotopy theory that there exists for any group G a space whose fundamental group is isomorphic to G and whose universal cover is contractible. Such a universal cover would then help us to construct a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ as described in the previous section. We turn in this section to a completely algebraic construction of such a resolution.

Let F_n be the free $\mathbb{Z}$-module on G^{n+1}, so a basis is given by all $(n+1)$-tuples $(g_0, g_1, \dots, g_n)$. We let G act on F_n by left multiplication componentwise, i.e., $g \cdot (g_0, g_1, \dots, g_n) = (gg_0, gg_1, \dots, gg_n)$. Each F_n is a free $\mathbb{Z}G$-module as one can choose $(1, g_1, \dots, g_n)$ as representatives of the G-orbits. It is often more convenient to use the following set of basis elements.

Definition 6.1. Let $n \geq 0$ and $g_1, \dots, g_n$ be elements of the group G. The $(n+1)$-tuple $(1, g_1, g_1g_2, \dots, g_1g_2\dots g_n)$ is written $[g_1|g_2|\dots|g_n]$ in the *bar notation*. When $n = 0$ there is a unique basis element $[\ ]$.

Remark 6.2. It seems that the bar resolution we describe next (also known as the the standard resolution) has been found independently by Beno Eckmann, a student of Heinz Hopf, in Zurich, and Samuel Eilenberg together with Saunders Mac Lane. This happened in the 1940's, during a time where communication was not easy. Freudenthal was also working on similar topics, developing tools to compute the homology of groups. It seems that Hopf and Freudenthal, both German, did not know about each other's work until after WWII. Heinz Hopf moved to Switzerland and asked for Swiss citizenship after his properties were confiscated by the Nazis in 1940. Hans Freudenthal meanwhile was in the Netherlands, which was occupied by the Germans. He lost his position in Amsterdam, and was even deported to a concentration camp in Havelte in 1943, from which he escaped in 1944 with help from his wife. He managed to hide in Amsterdam till the end of the war.

We construct now a chain complex by using standard simplicial technology. We write

$$\partial_i(g_0, g_1, \dots, g_n) = (g_0, \dots, \hat{g}_i, \dots, g_n)$$

for the n-uple obtained by forgetting the i-th entry g_i, and construct then a differential $d\colon F_n \to F_{n-1}$ by setting $d = \sum(-1)^i\partial_i$. This is an example of chain complex one obtains from a so-called *simplicial object*, just like the Hochschild complex you have met in the exercises.

Lemma 6.3. *The chain complex $F_\bullet$ is a chain complex of free $\mathbb{Z}G$-modules. It provides a free resolution of $\mathbb{Z}$ via the augmentation map $\varepsilon\colon F_0 = \mathbb{Z}G \to \mathbb{Z}$.*

The differentials are completely determined by their effect on the elements written in the "bar notation". We compute $\partial_i[g_1|\dots|g_n] = \partial_i(1, g_1, g_1g_2, \dots, g_1g_2\dots g_n)$

$$\partial_i[g_1|\dots|g_n] = \begin{cases} g_1(1, g_2, \dots, g_2\dots g_n) & \text{if } i = 0 \\ (1, g_1, \dots, g_1\dots g_{i-1}, g_1\dots g_{i+1}, \dots, g_1g_2\dots g_n) & \text{if } 0 < i < n \\ (1, g_1, g_1g_2, \dots, g_1g_2\dots g_{n-1}) & \text{if } i = n \end{cases}$$

In the bar notation this gives

$$\partial_i[g_1|\dots|g_n] = \begin{cases} g_1[g_2|\dots|g_n] & \text{if } i = 0 \\ [g_1|\dots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\dots|g_n] & \text{if } 0 < i < n \\ [g_1|\dots|g_{n-1}] & \text{if } i = n \end{cases}$$

7. Classifying spaces

We will construct a space from this resolution by building the geometric realization of the simplicial object given by G^{n+1} in degree n (before linearizing to get the bar resolution). We want the cellular chain complex of this space to coincide with the bar resolution, so we need to have an n-cell for each $(n+1)$-tuple $\sigma = (g_0, \dots, g_n)$. In order to respect the geometry of simplices the n-cell we use for σ is a standard n-simplex $\Delta_\sigma = \Delta_n$. Beware that this space will not be a simplicial complex, the vertices do not determine one simplex as they appear with an order in G^{n+1}. For example, in the bar resolution for the group $C_2 = \{1, t\}$ we have two 1-cells corresponding to $(1, t)$ and $(t, 1)$. As we will see right now these two edges have common vertices 1 and t.

The space EG is a quotient of a disjoint union of Δ_σ's taken over all $\sigma \in G^{n+1}$, for all $n \geq 0$. The identification we make should correspond to the differential we have defined above. Therefore we identify the $(n-1)$-simplex $\Delta_{\partial_i\sigma}$ with its image as the i-th face of Δ_σ. This means that EG is a CW-complex with as many 0-cells as elements in G, as many 1-cells as pairs in G^2, etc. The identifications for every (ordered) pair (g_0, g_1), corresponding to a 1-simplex, glues its end points to the 0-cells corresponding to g_1 and g_0 since its faces should be identified with the 0-cells corresponding to $\partial_0(g_0, g_1) = g_1$ and g_0 respectively. We continue with 2-cells and so on.

Example 7.1. Let $G = C_2 = \langle t \mid t^2 \rangle$. The CW-complex EC_2 consists of two 0-cells corresponding to 1 and t, four 1-cells corresponding to $(1, t)$ and $(t, 1)$ whose end points are 1 and t, as well as $(1, 1)$ and (t, t) creating loops at 1 and t respectively. The latter two, with repeated consecutive elements, are called *degenerate*. Among the eight 2-cells only two are non-degenerate, namely $(1, t, 1)$ and $(t, 1, t)$. At this stage we recognize a copy of S^2 given by the non-degenerate cells of dimension ≤ 2, and in the end EC_2 looks like a fat version of S^∞ with many more degenerate cells than the two we need in the geometric model constructed previously.

From the point of view of group homology, the following proposition is not relevant, but it allows us to connect the bar resolution with the theory of covering spaces. We choose not to spell out all the technical details of the proof.

Proposition 7.2. *The space EG is contractible and its cellular chain complex is isomorphic to the bar resolution.*

Proof. The second part is clear by design, but the acyclicity of the cellular chain complex is not enough to conclude that EG is contractible. We have thus to find an analogous argument to the one we performed to prove the contractibility of the augmented bar resolution.

For each n-dimensional simplex $\sigma = (g_0, \dots, g_n)$ consider the $(n+1)$-simplex $h\sigma = (1, g_0, \dots, g_n)$. The 0-th face inclusion $\partial_0 \colon \Delta_\sigma \hookrightarrow \Delta_{h\sigma}$ is homotopic to the constant map at the 0-th vertex via an affine homotopy (every point in the 0-face moves to the 0-th vertex on a straight line). We run all these homotopies simultaneously, for all σ, which shows that EG contracts onto the 0-cell corresponding to $1 \in G$. $\square$

Definition 7.3. Let G be a group and EG a contractible CW-complex on which G acts by freely permutating the cells. We call the quotient $BG = EG/G$ a *classifying space* of G.

The space EG is not unique, we will use the model described above and based on the bar resolution. The quotient space is thus not uniquely determined (up to homeomorphism), but it is so up to homotopy.

Theorem 7.4. *The quotient map $EG \to BG$ is a universal cover of the classifying space.*

PROOF. Since the action of G permutes freely the cells of the CW-complex EG, the quotient map is a covering map (even a Galois covering). We also know that the fundamental group $\pi_1 EG$ is trivial by contractibility of EG. Therefore it is the universal cover of BG. □

For those who have already met higher homotopy groups, notice that $\pi_n BG \cong \pi_n EG = 0$ for all $n \geq 2$ by the homotopy lifting property of a covering map. This means that BG is an *Eilenberg-Mac Lane space* of type $K(G, 1)$, having a unique non-trivial homotopy group, namely the first one, isomorphic to G. The homotopy type of such CW-complexes is entirely determined by the group G.

We conclude this section by observing that group homology is a particular instance of singular or cellular homology. Singular homology groups are homotopy invariants, we can thus use any model for BG. We will again stick to our construction explained above.

Corollary 7.5. *Let G be a group. Then $H_n(G; \mathbb{Z}) \cong H_n(BG; \mathbb{Z})$.*

PROOF. Singular homology and cellular homology coincide on CW-complexes. We compute therefore the homology of BG from the cellular chain complex of BG. As G permutes freely the cells of EG, this chain complex is isomorphic to the coinvariants of the bar resolution, which computes the homology of the group G. □

8. Homology groups in small degree

After this topological parenthesis we come back to group homology and try to understand the meaning of these invariants in low degrees. The zeroth left derived functor of a right exact functor gives us back this very same functor. This means here that $H_0(G; \mathbb{Z})$ is $\mathbb{Z}_G$, the coinvariants of a trivial G-module, hence $H_0(G; \mathbb{Z}) \cong \mathbb{Z}$ for any group G.

Let us check how this shows up in the bar resolutions, and let us compute at the same time the first homology group $H_1(G; \mathbb{Z})$. The bar resolution starts with free abelian groups on triples, pairs, and elements of G:

$$\mathbb{Z}[G \times G \times G] \xrightarrow{d_2} \mathbb{Z}[G \times G] \xrightarrow{d_1} \mathbb{Z}[G]$$

Taking coinvariants under the left action of G leaves us with

$$\mathbb{Z}[G \times G] \xrightarrow{d_2} \mathbb{Z}[G] \xrightarrow{d_1} \mathbb{Z}$$

The formulas at the end of Section 6 provide an explicit description of the two differentials, expressed on generators written in the bar notation. In the bar resolution we have $d_1[g] = g[\,] - [\,]$, so $d_1 = 0$ on coinvariants. This recovers the computation of $H_0(G;\mathbb{Z}) \cong \mathbb{Z}$.

We identify next d_2. We have $d_2[g|h] = g[h] - [gh] + [g]$, so on coinvariants the formula $d_2[g|h] = [g] + [h] - [gh]$ for generators determines d_2.

Proposition 8.1. *Let G be a group. Then $H_1(G;\mathbb{Z}) \cong G_{ab}$.*

Proof. By definition $H_1(G;\mathbb{Z})$ is isomorphic to $\mathbb{Z}[G]$ divided out by the image of d_2. Construct a map $G \to \mathbb{Z}[G]$ by sending g to $[g]$. Postcomposing with the quotient map to $\mathbb{Z}[G]/\operatorname{Im} d_2$ becomes a group homomorphism, from the multiplicative group G to the abelian group $H_1(G;\mathbb{Z})$ (written additively). We verify that this surjective homomorphism satisfies the universal property of abelianization. For any group homomorphism $\alpha\colon G \to A$ to an abelian group A, let us construct a group homomorphism $\mathbb{Z}[G] \to A$ by sending the generator $[g]$ to $\alpha(g)$. This homomorphism is compatible with the relations $[g] + [h] - [gh] = 0$ because α is a homomorphism. It induces therefore a homomorphism $H_1(G;\mathbb{Z}) \to A$ which is the unique homomorphism we are looking for. □

9. The Hopf formula

We conclude this chapter on homology groups with a classical computation of Heinz Hopf. It shows the kind of information higher homology groups contain, maybe also the fact that one should not hope for nice and clean formulas for all homology groups, as the intricacy increases along with the degree.

We start with a purely homological observation. Let $C_\bullet$ be a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ and let us call $T_nC_\bullet$ the truncated subcomplex defined by $(T_nC)_k = 0$ for $k > n$

and $(T_nC)_k = C_k$ for $k \leq n$. We call $C(n)_\bullet$ the quotient $C_\bullet/T_nC_\bullet$ in analogy with the notation used for n-connected covers (even though the map goes the wrong way around).

Lemma 9.1. *For all $k < n$ we have $H_k((T_nC_\bullet)_G) \cong H_k(G;\mathbb{Z})$ and there is a four term exact sequence*

$$0 \to H_{n+1}(G;\mathbb{Z}) \to H_n(T_nC_\bullet)_G \to H_n((T_nC_\bullet)_G) \to H_n(G;\mathbb{Z}) \to 0$$

Proof. The isomorphism in low degrees is clear since the truncated complex coincides with the free resolution in degrees $\leq n$. For the second part we consider the short exact of chain complexes $0 \to T_nC_\bullet \to C_\bullet \to C(n)_\bullet \to 0$. Taking coinvariants yields another short exact sequence of complexes, not because coinvariants is an exact functor, it is not, but because in each degree the exact sequence is either an isomorphism followed by the zero map or the other way around.

There is an associated long exact sequence whose relevant part is the following

$$0 \to H_{n+1}((C_\bullet)_G) \to H_{n+1}((C(n)_\bullet)_G) \xrightarrow{\partial} H_n((T_nC_\bullet)_G) \to H_n((C_\bullet)_G) \to 0$$

The sequence starts with zero because $(T_nC_\bullet)_G$ is concentrated in degrees $\leq n$ and ends with zero because $(C(n)_\bullet)_G$ is concentrated in degrees $> n$. The first and last group are homology groups of G by definition (homology groups of the coinvariants applied to a free resolution). It remains thus to identify the second and third groups.

We see that $H_{n+1}((C(n)_\bullet)_G) = \mathrm{Coker}((d_{n+2})_G)$. But coinvariants is a right exact functor, so this is isomorphic to the cokernel of the differential $d_{n+2}\colon C_{n+2} \to C_{n+1}$ of which we take the coinvariants. By exactness of the resolution, this cokernel is isomorphic to the image of d_{n+1}, i.e. the kernel of d_n. All together we have identified this homology group with $H_n(T_nC_\bullet)_G$. The third homology group is already in the form we desire. Let us maybe just add that this n-th homology group of a truncated complex is the kernel of the coinvariants differential $(d_n)_G$. □

There are purely algebraic proofs of the Hopf formula, but we prefer to follow Brown's strategy and mix it with some covering space theory.

Proposition 9.2. *Let $\langle S \mid R\rangle$ be a presentation of a group G, $F = F(S)$ be the free group on S and N the normal subgroup of F generated by the relations R. There is* *a four term exact sequence*

$$0 \to H_2(G;\mathbb{Z}) \to H_1(N;\mathbb{Z})_G \to H_1(F;\mathbb{Z}) \to H_1(G;\mathbb{Z}) \to 0$$

PROOF. Let Y be a wedge of S circles, so that $\pi_1 Y \cong F(S) = F$ and let E be the (Galois) covering corresponding to the normal subgroup N. The Galois group of deck transformations is isomorphic to G, it acts freely on the 0- and 1-cells of the 1-dimensional CW-complex E. Therefore the cellular chain complex $C_\bullet^{cell}(E)$ forms a truncated free resolution of the trivial module $\mathbb{Z}$. We conclude from Lemma 9.1 that $H_2(G;\mathbb{Z})$ is isomorphic to the kernel of the map $H_1(E;\mathbb{Z})_G \to H_1(E/G;\mathbb{Z}) \cong F(S)_{ab}$.

Our aim is therefore to identify the coinvariants of $H_1(E;\mathbb{Z}) \cong \pi_1(E)_{ab} = N_{ab}$. We claim that the Galois action coincides with the conjugation action: any element of G can be represented by a word $w \in F(S)$ and N is a normal subgroup of $F(S)$. Then wnw^{-1} belongs to N and the class in the abelianization does not depend on the choice of w since conjugation by an element of N is trivial in N_{ab}.

Now that this action is defined we enhance the isomorphism $N_{ab} \cong H_1(E;\mathbb{Z})$ with a G-equivariant structure by making the map $N_{ab} \to H_1(E;\mathbb{Z})$ more explicit. We start from $N \cong \pi_1 E$. Given $n \in N$, seen as a subgroup of $F(S)$, we represent n by a loop in Y and lift it to a loop ν in E, based at a vertex e. Next, for any $w \in F(S)$, the conjugate wnw^{-1} lifts to the concatenation of a path from e to $w(e)$ (remember the monodromy action on the fiber over the base point of Y), followed by the translated loop $\overline{w}\nu$ under the action of $\overline{w} \in G$, and coming finally back to e by the inverse path going from $w(e)$ to e. The image in homology of this loop based at e is the corresponding cycle in $H_1(E;\mathbb{Z})$. Homology is an unpointed functor, this cycle has the same image as $\overline{w}\nu$, which shows that the map is compatible with the action of G. □

Theorem 9.3. *(Hopf Formula) Let $\langle S \mid R\rangle$ be a presentation of a group G, $F = F(S)$ be the free group on S and N the normal subgroup of F generated by the relations R. Then $H_2(G;\mathbb{Z}) \cong N \cap [F,F]/[F,N]$.*

PROOF. We are ready to come back to the description of the second homology group. We have understood now that it is isomorphic to the kernel of the map $(N_{ab})_G \to F(S)_{ab}$. By definition of the conjugation action of G on N_{ab} introduced above, we see that $(N_{ab})_G = (N/[N,N])_G \cong N/[F,N]$ since we have to identify wnw^{-1} with n for all $w \in F$ and $n \in N$. We conclude finally that $H_2(G;\mathbb{Z})$ is isomorphic to $\mathrm{Ker}(N/[F,N] \to F/[F,F]) = N \cap [F,F]/[F,N]$. □

The Hopf formula is difficult to use in practice as it involves the computation of mixed commutators $[F,N]$ in a free group, we will illustrate its usefulness in exercises. What we should also remember is that the Hopf formula gives a relation between the second homology of a quotient in terms of the other ingredients in an extension (namely F and N). We will come back to such extensions when we study the second cohomology group of a group.

Remark 9.4. We have already met Heinz Hopf in the previous section, for his early work on group homology. In Zurich he founded the Swiss school in algebraic topology and had an enormous impact in mathematics, also through his student Beno Eckmann who founded the FIM at ETHZ. Heinz Hopf (1894-1971) received his PhD in 1925 under Ludwig Bieberbach, notorious Nazi, whereas Hopf had a Jewish father. The Hopf formula appears in a paper in 1942. Bieberbach was dismissed form his academic position in 1945, but in 1949 was invited to lecture at the University of Basel by Ostrowski, who considered Bieberbach's political views irrelevant to his contributions to the field of mathematics (Wikipedia).

10. Homology with arbitrary coefficients

To compute the homology groups $H_n(G;\mathbb{Z})$ with coefficients in the trivial G-module $\mathbb{Z}$ we left derive the functor $-\otimes_{\mathbb{Z}G}\mathbb{Z}$ evaluated at $\mathbb{Z}$. This means we take a free $\mathbb{Z}G$-resolution $F_\bullet$ of $\mathbb{Z}$, compute coinvariants $(F_\bullet)_G$ and take homology.

Remark 10.1. To write the tensor product as above we should have a right G-action on $F_\bullet$, but we said that we would take a resolution of left G-modules. To solve this issue, notice that any left G-module M can be seen as a right module by setting $g \cdot m = mg^{-1}$.

Let us take the opportunity to make a more general observation. The tensor product $M \otimes_G N = (M \otimes N)_G$ coincides with the coinvariants of the tensor product where G acts diagonally $g(m \otimes n) = gm \otimes gn$. Indeed, when forming the tensor product over the group ring $\mathbb{Z}G$ we identify $mg \otimes n = g^{-1}m \otimes n$ with $m \otimes gn$. When $m = gm$ we get exactly that $gm \otimes gn = m \otimes n$ in the quotient.

We generalize now this definition to arbitrary G-modules.

Definition 10.2. Let $F_\bullet$ be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and M be a G-module. The *homology of* G *with coefficients in* M is $H_*(G; M) = H_*(F_\bullet \otimes_G M)$.

Example 10.3. Because the tensor product is right exact, we have $H_0(G; M) = H_0(F_\bullet \otimes_G M) \cong \mathbb{Z} \otimes_G M \cong M_G$. The zeroth homology group coincides with the coinvariants.

Example 10.4. When $M = \mathbb{Z}G$ is the group ring itself, seen as a left G-module, then $(\mathbb{Z}G)_G = \mathbb{Z}$ as we know, but $F_\bullet \otimes_G \mathbb{Z}G \cong F_\bullet$, so $H_k(G; \mathbb{Z}G) = 0$ for all $k > 0$.

CHAPTER 2

Group cohomology

This second chapter dualizes the approach taken in the first one. The unexpected bonus of this is that cohomology groups come with more structure, that of a graded ring. We follow Brown's book [**7**] for large parts here as well.

1. Cohomology groups

Whereas homology groups are related to the tensor product, cohomology groups are defined via Hom. Choose a projective resolution $F_\bullet$ (free for example) of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, consider the cochain complex $\mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M)$ for some left $\mathbb{Z}G$-module M.

Definition 1.1. Let G be a group and M a left $\mathbb{Z}G$-module. The n-th *cohomology group* $H^n(G; M)$ is the n-th cohomology group of the cochain complex $\mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M)$.

This means that we use as differential $\mathrm{Hom}_{\mathbb{Z}G}(d_n, M)$ to compute the right derived functors of Hom, which are Ext groups. Brown uses the same differential up to a sign $(-1)^n$ because he considers this cochain complex as a particular case of a chain complex $\mathscr{H}om_{\mathbb{Z}G}(F_\bullet, F'_\bullet)$ where $F'_\bullet$ could be any chain complex (as seen in the exercise sheet of Week 3), and for us it is just a single module concentrated in degree zero.

We identify now the zeroth cohomology group. In homology we saw that coinvariants appeared, here we will deal with invariants.

Lemma 1.2. *Let G be a group and M a left $\mathbb{Z}G$-module. The 0-th cohomology group $H^0(G; M)$ is isomorphic to the invariants M^G.*

Proof. The functor $\mathrm{Hom}_{\mathbb{Z}G}(-M)$ is left exact, so that the exact sequence

$$F_1 \to F_0 \to \mathbb{Z} \to 0$$

induces an exact sequence $0 \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \to \mathrm{Hom}_{\mathbb{Z}G}(F_0, M) \xrightarrow{d^1} \mathrm{Hom}_{\mathbb{Z}G}(F_1, M)$. The zeroth cohomology group is by definition the kernel of d^1, which coincides with $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \cong M^G$. Indeed, any homomorphism of $\mathbb{Z}G$-modules from the trivial module $\mathbb{Z}$ must land in the submodule of G-invariant elements, and the image of $1 \in \mathbb{Z}$ determines any such homomorphism uniquely. □

Example 1.3. Let C be the infinite cyclic group and $0 \to \mathbb{Z}C \xrightarrow{t-1} \mathbb{Z}C \xrightarrow{\varepsilon} \mathbb{Z}$ the free resolution $F_\bullet$ from Example 2.3 in Chapter 1. We compute the cohomology with coefficients in any module M. The cochain complex $\mathrm{Hom}_{\mathbb{Z}C}(F_\bullet, M)$ is

$$0 \to M \xrightarrow{t-1} M \to 0.$$

Therefore $H^0(C; M) \cong M^C$ as we already know from Lemma 1.2 and $H^1(C; M) \cong M_C$. All higher cohomology groups are trivial.

Example 1.4. Let C_2 be the cyclic group of order 2 and consider the free resolution $F_\bullet$

$$\dots \xrightarrow{t-1} \mathbb{Z}C_2 \xrightarrow{t+1} \mathbb{Z}C_2 \xrightarrow{t-1} \mathbb{Z}C_2 \xrightarrow{\varepsilon} \mathbb{Z}$$

from Exercise 1, Week 1. We compute the cohomology with coefficients in any module M. The cochain complex $\mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M)$ is

$$0 \to M \xrightarrow{t-1} M \xrightarrow{t+1} M \xrightarrow{t-1} \dots$$

Therefore $H^0(C; M) \cong M^C$ as we already know and from there on there is a periodicity. All even cohomology groups $H^{2n}(C_2; M) \cong M^{C_2}/(t+1)M$ and odd ones $H^{2n+1}(C_2; M)$ are given by $\mathrm{Ker}(t+1)/\mathrm{Im}(t-1)$, which can be identified with a subgroup of the coinvariants, see Exercise 2 of Week 3.

For example, when $M = \mathbb{F}_2$, a trivial module, both maps $t-1$ and $t+1$ coincide and are trivial. Therefore $H^n(C_2; \mathbb{F}_2) \cong \mathbb{F}_2$ for all $n \geq 0$.

2. The first cohomology group

Just like we did for homology groups, let us use the bar resolution to identify the first cohomology group, this time with coefficients in any $\mathbb{Z}G$-module. We need therefore to consider the bar resolution up to degree 2, and take homomorphisms into a $\mathbb{Z}G$-module M. Since $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M) \cong M$, the cochain complex we obtain looks like this:

$$0 \to M \xrightarrow{d^1} \mathrm{Fun}(G, M) \xrightarrow{d^2} \mathrm{Fun}(G^2, M)$$

where we have identified $\mathbb{Z}G$-module homomorphisms out of a free module with basis $[g]$, respectively $[g|h]$ with functions out of this basis.

Let us next identify the coboundary maps. In the bar resolution $d_1[g] = g[] - []$ and $d_2[g|h] = g[h] + [g] - [gh]$ as we have seen in the previous chapter. Therefore $d^1(m)$ is the function that takes g to $gm - m$ and for any function $f\colon G \to M$

$$d^2(f)(g, h) = gf(h) + f(g) - f(gh)$$

Definition 2.1. Let G be a group and M a $\mathbb{Z}G$-module. A *derivation* is a function $d\colon G \to M$ such that $d(gh) = gd(h) + d(g)$. A *principal derivation* is a derivation of the form $d(g) = gm - m$ for some fixed element $m \in M$. We write $\mathrm{Der}(G, M)$ for the group of derivations and $\mathrm{Prin}(G, M)$ for the subgroup of principal derivations.

It is easy to check that a principal derivation is a derivation since $g(hm - m) + (gm - m) = gh \cdot m - m$. The above computation with the bar resolution shows that the first cohomology group of G measures the difference between derivations and principal ones.

Proposition 2.2. *Let G be a group and M a $\mathbb{Z}G$-module. There is an isomorphism $H^1(G; M) \cong \mathrm{Der}(G, M)/\,\mathrm{Prin}(G, M)$.* □

In the particular case where M is a trivial $\mathbb{Z}G$-module, we can identify derivations and principal ones more explicitly.

Corollary 2.3. *Let G be a group and M a trivial $\mathbb{Z}G$-module. There is an isomorphism $H^1(G; M) \cong \mathrm{Hom}(G_{ab}, M)$.*

Proof. Principal derivations are all trivial since $gm = m$ and a derivation d satisfies in this particular case that $d(gh) = d(h) + d(g)$, in other words d is a homomorphism from G to the abelian group M. Such a homomorphism factors uniquely through the abelianization G_{ab}. □

3. The first cohomology group and split extensions

We move now to extensions and will use the previous computation to classify splittings. We first fix the notation and remind the terminology. An extension of a group G by another group N is a short exact sequence $1 \to N \xrightarrow{i} E \xrightarrow{\pi} G \to 1$. Sometimes we will say abusively that the extension is E. Two extensions E and E' are equivalent if there is an isomorphism $f\colon E \to E'$ that is compatible with the inclusion of the normal subgroup N and the projection to the quotient G, i.e., $f \circ i = i'$ and $\pi' \circ f = \pi$.

In this section we consider extensions with abelian kernel, let us write it suggestively M since there is a conjugation action of G on M making it a $\mathbb{Z}G$-module. For an element $g \in G$, choose a preimage $e \in E$ such that $\pi(e) = g$ and define $g \cdot m$ to be the unique element in M whose image under the inclusion i is $ei(m)e^{-1}$. This belongs to M which is a normal subgroup of E and is well defined because M is abelian (the conjugation action of M on itself is trival).

Definition 3.1. Let E be an extension of G by an abelian group M. If the action of G on M is trivial we call the extension a *central extension*.

The meaning is probably transparent: in a central extension the kernel M is contained in the center of E.

We fix now a group G and an arbitrary $\mathbb{Z}G$-module M. Our aim is to understand the easiest extensions of G by M, giving rise to the given module structure on the kernel M, namely the split extensions.

Definition 3.2. Let E be an extension of G by $\mathbb{Z}G$-module M. We say that it is a *split extension* if there exists a *section* $s\colon G \to E$. This section is a group homomorphism such that $\pi \circ s = \mathrm{id}_G$.

The classification problem of split extensions of a given group by a given $\mathbb{Z}G$-module M is boring, there is a unique one (see also Exercise 4 of Week 4).

Proposition 3.3. *An extension E of G by M is split if and only if E is equivalent to the semi-direct product extension $M \rtimes G$.* □

We recall that the semi-direct product is $M \times G$ as a set, equipped with the group law given by the formula

$$(m, g) \cdot (n, h) = (m + gn, gh)$$

for all $m, n \in M$ and $g, h \in G$.

The classification of split extensions up to equivalence is done, but let us look at possibly different splittings. How many of them exist?

Definition 3.4. Let E be a split extension of G by a $\mathbb{Z}G$-module M. Two splittings s, s' are *conjugate* if there is an element $m \in M$ such that $s(g) = i(m)s'(g)i(m)^{-1}$.

In order to classify sections up to conjugacy for any split extension, we use Proposition 3.3 and can assume that E is a semi-direct product. We must thus study sections $s\colon G \to M \rtimes G$.

Proposition 3.5. *Let G be a group and M a $\mathbb{Z}G$-module. The conjugacy classes of sections of the semi-direct product $M \rtimes G$ are in bijection with the elements of $H^1(G; M)$.*

PROOF. A section $s\colon G \to M \rtimes G$ must have the form $s(g) = (dg, g)$ where $d\colon G \to M$ is a function. But not just any function because s must be a group homomorphism, which translates as:

$$(d(gh), gh) = (dg, g)(dh, h) = (dg + gdh, gh)$$

so that d is a derivation. Now we have identified all sections, they correspond precisely to derivations. Let us look next at what it means for two sections to be conjugate. Two such sections correspond to two derivations d and d'. If there is an element $m \in M$ such that $s(g) = i(m)s'(g)i(m)^{-1}$, this becomes here

$$(dg, g) = (m, 1)(d'g, g)(m, 1)^{-1} = (m + d'g, g)(-m, 1) = (m + d'g + g(-m), g)$$

Therefore the difference $d' - d$ is the principal derivation corresponding to m. The proposition is proven thanks to our computation of the first cohomology group in Proposition 2.2. $\square$

4. The classification of extensions with abelian kernel

We have seen in the previous section that split extensions of a group G by an abelian group M are uniquely determined, up to equivalence of extensions, by the $\mathbb{Z}G$-module structure on M. Our aim in this section is to classify all extensions of G by a given $\mathbb{Z}G$-module M. Let us thus consider an extension

$$0 \to M \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

Even if this extension does not split we can choose a set theoretical section $s\colon G \to E$ and we might as well assume that $s(1_G) = 1_E$ (this is a so-called normalization assumption). In general, unless the extension splits, this section is not a homomorphism. For any $g, h \in G$, we measure the difference between $s(gh)$ and $s(g)s(h)$ as follows. Since $\pi(s(gh)) = gh = \pi(s(g))\pi(s(h))$ the product $s(g)s(h)s(gh)^{-1}$ belongs to the kernel of π, or equivalently to the image of i. There exists therefore an element $f(g,h) \in M$ such that

$$\boxed{i[f(g,h)]s(gh) = s(g)s(h)}$$

We call the function f the *factor set* associated to the section s. We remark that when g or h is equal to 1 normalization implies that $f(1,h) = 0 = f(g,1)$ for any $g, h \in G$.

Lemma 4.1. *Let E be an extension of a group G by a $\mathbb{Z}G$-module M. Any factor set $f\colon G \times G \to M$ allows us to recover the extension up to equivalence.*

Proof. We start with the observation that the map $M \times G \to E$ given by $(m, g) \mapsto i(m)s(g)$ is a bijection (of sets). To recover the group law on $M \times G$ we compute the product for two pairs (m, g) and (n, h):

$$i(m)s(g)i(n)s(h) = i(m)s(g)i(n)s(g)^{-1}s(g)s(h) = i(m)i(gn)s(g)s(h)$$

where we used the fact that G acts on M by conjugation in E. But i is a group homomorphism and we can use the defining property of the factor set to obtain

$$i(m)s(g)i(n)s(h) = i(m + gn)i(f(g,h))s(gh) = i(m + gn + f(g,h))s(gh)$$

Therefore the group law on the set $M \times G$ is given by the formula

$$(m, g)(n, h) = (m + gn + f(g,h), gh)$$

We write $M \times_f G$ for the set $M \times G$ equipped with this composition law (which we have not checked yet is a group). However the composition law has been designed in such a way that the map $\varphi \colon M \times_f G \to E$ defined by the obvious formula $\varphi(m, g) = i(m)s(g)$ is compatible with the group structure on E. Therefore the element $(0, 1)$ is a neutral element in $M \times_f G$, the inverse in E for $i(m)s(g)$ provides an inverse for (m, g), and the composition is associative since so is the group law on E. We will come back to this in the next step, by solving explicitly these equations corresponding to inversion and associativity.

Before that, we conclude by noting that we have just defined an equivalence φ between two group extensions of G by the $\mathbb{Z}G$-module M. It is compatible with the inclusion $m \mapsto (m, 1)$ since $s(1) = 1$ by normalization assumption and with the projection $(m, g) \mapsto g$ since $\pi(i(m)s(g)) = g$. □

Let us look more closely at the group structure defined in the previous proof by means of a function $f \colon G \times G \to M$. As we have already observed the element $(0, 1)$ is a two-sided neutral element if for all $g, h \in G$:

$$\boxed{f(1, h) = 0 = f(g, 1)}$$

Second we wonder if any element (m, g) has an inverse, so we must solve the equation

$$(0, 1) = (m, g)(n, h) = (m + gn + f(g, h), gh)$$

Thus $h = g^{-1}$ and $m + gn + f(g, g^{-1}) = 0$. Hence $n = -g^{-1}(m + f(g, g^{-1}))$. This gives us a right inverse, which has to be also a left inverse if the composition law is associative, something we have to verify anyway.

Thirdly, we compute a triple product

$$[(m, g)(n, h)](p, k) = (m+gn+f(g, h), gh)(p, k) = (m+gn+f(g, h)+ghp+f(gh, k), ghk)$$

And likewise

$$(m, g)[(n, h)(p, k)] = (m, g)(n+hp+f(h, k), hk) = (m+g(n+hp+f(h, k))+f(g, hk), ghk)$$

By comparing the first coordinate we see that we must require for all $g, h, k \in G$:

$$\boxed{f(g, h) + f(gh, k) = gf(h, k)) + f(g, hk)}$$

Proposition 4.2. *There is a bijection between equivalence classes of extensions of a group G by a $\mathbb{Z}G$-module M equipped with a normalized section and functions $f\colon G \times G \to M$ satisfying the two boxed equalities.*

Proof. We have started the section by constructing a factor set from a normalized section. This factor set must satisfy the two properties since $M \times_f G$ is a group as we have seen in Lemma 4.1. Conversely, given such a function $f\colon G \times G \to M$, we have just explained that the construction $M \times_f G$ yields a group. The inclusion $M \to M \times_f G$ sending m to $(m, 1)$ is a group homomorphism since $(m, 1)(n, 1) = (m + 1 \cdot n, 1 \cdot 1) = (m + n, 1)$, and the projection $(m, g) \mapsto g$ is obviously also a group homomorphism. This presents $M \times_f G$ as an extension of G by the abelian group M, so we only need to verify that the $\mathbb{Z}G$-module structure agrees with the given one.

The action of G on M is by conjugation in the extension $M \times_f G$. Let us choose $(0, g)$ as our favorite preimage, and recall from above that its inverse is $(-g^{-1}f(g, g^{-1}), g^{-1})$. Compute then $(0, g)(m, 1)(-g^{-1}f(g, g^{-1}), g^{-1})$:

$$(gm+f(g,1),g)(-g^{-1}f(g,g^{-1}),g^{-1}) = (gm-gg^{-1}f(g,g^{-1})+f(g,g^{-1}),gg^{-1}) = (gm,1)$$

Finally we also recover the normalized section by setting $s(g) = (0, g)$. □

We are ready to prove the main result of this section.

Theorem 4.3. *There is a bijection between equivalence classes of extensions of a group G by a $\mathbb{Z}G$-module M and $H^2(G; M)$.*

Proof. We continue the argument that we started in Proposition 4.2. We view the factor set f as a cochain in $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}[G^3], M)$ via the identification of the generators in the bar notation. The second boxed equality can be rewritten as

$$gf(h,k)) - f(gh,k) + f(g,hk) - f(g,h) = 0$$

which we recognize as a 2-cocycle condition. In this way we have associated to our original extension a 2-cocycle, representing a class in $H^2(G; M)$. We have made choices (a normalized section, leading us to consider special cocycles satisfying $f(g, 1) = 0 = f(1, h)$) and we need to check that they do not affect the cohomology class.

Another section $s'\colon G \to E$ can be obtained from s by mutiplying s by a function $c\colon G \to M$ as follows: $s'(g) = i(c(g))s(g)$. Observe that $p[s(1)] = 1$ so $s(1) \in \operatorname{Ker} p = \operatorname{Im} i$, which we identify with M. By choosing the function c such that $c(1) = -s(1)$ in M, we obtain thus a normalized section s'. We will show that the 2-cocycle f' associated to s' differs from f precisely by dc, where we see c as a 1-cochain. Let us redo the computation of the factor set

$$s'(g)s'(h) = i(c(g))s(g)i(c(h))s(h) = i(c(g))i(gc(h))s(g)s(h)$$

because conjugation by a preimage $s(g)$ of g in E coincides with the action by g on M. On the right hand side we use now the fact that i is a group homomorphism, and the formula we obtained for the factor set f:

$$i(c(g) + gc(h))i(f(g,h))s(gh) = i(c(g) + gc(h) + f(g,h))s(gh)$$

We wish to see $s'(gh)$ on the right, so we introduce it also in the previous expression as an inverse: $s(gh) = [i(c(gh)]^{-1}s'(gh)$. All together we have therefore

$$s'(g)s'(h) = i(c(g) + gc(h) + f(g,h) - c(gh))s'(gh)$$

The difference $f'(g,h) - f(g,h) = gc(h) - c(gh) + c(g) = d^2c(g,h)$, a coboundary as claimed.

The last issue is the normalized form of the 2-cocycle. Brown solves this by using not the bar resolution, but a smaller, normalized form. We will do that in the exercises. □

We give a corollary about groups of order mn where $(m,n) = 1$. It is worth saying that the zero element in $H^2(G;M)$ classifies the split extension $M \rtimes G$. In this case the section can be chosen to be a group homomorphism and so the 2-cocycle f is zero.

Corollary 4.4. *Let m and n be coprime natural integers and assume that E is a group of order mn containing an abelian normal subgroup M of order m. Then E is isomorphic to a semi-direct product $M \rtimes G$ for a group G of order n and all subgroups of E of order n are conjugate.*

Proof. Let $G = E/M$ and consider the extension $0 \to M \to E \to G \to 1$. Equivalence classes of such extensions are classified by $H^2(G; M)$ by Theorem 4.3. We will show in the exercises that this cohomology group is zero because the order of G is invertible in M.

If we assume for a moment that this is true, then we conclude that the only extension of G by the $\mathbb{Z}G$-module M splits. The number of sections up to conjugacy is given by the order of $H^1(G; M)$, which is zero for the same reasons as $H^2(G; M)$. We conclude by the fact that any subgroup $H < E$ of order n maps isomorphically onto G since m and n are coprime. □

5. The cross product

Before introducing the cup product on the cohomology of a single group we define an operation on the homology and cohomology of two different groups taking value in the (co)homology of their product. In this section G and G' are two groups, M and M' are modules over the respective group rings.

We observe first that $M \otimes M'$ is a module over $\mathbb{Z}(G \times G')$, and since $\mathbb{Z}G \otimes \mathbb{Z}G' \cong \mathbb{Z}(G \times G')$, a tensor product of free modules is free over $\mathbb{Z}(G \times G')$. Before proving that a tensor product of resolutions is again a resolution, we need a lemma in homological algebra. Recall that in general contractibility is a stronger notion than acyclicity for chain complexes (typically a chain complex such as $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$, given by a non-split extension, is not contractible).

Lemma 5.1. *Let $F_\bullet$ be an acyclic resolution of free $\mathbb{Z}$-modules. Then $F_\bullet$ is contractible.*

Proof. Since F_n is free for all n, so is the image $\operatorname{Im} d_{n+1} = A_{n+1} \subset F_n$. By exactness, F_n is an extension of $\operatorname{Im} d_n = A_n$ by $\operatorname{Im} d_{n+1}$, which must split, choose a section $s_n\colon A_n \to F_n$.

Let $B_n = \operatorname{Im} s_n \subset F_n$ and observe that it is isomorphic to $A_n = \operatorname{Im} d_n$ so that $F_n = A_{n+1} \oplus B_n$. We define now a contracting homotopy h as follows. We choose $h_n\colon F_n \to F_{n+1}$ to be the zero map on B_n and on A_{n+1} the inverse s_{n+1} of the isomorphism $d_{n+1}|_{B_{n+1}}\colon B_{n+1} \to \operatorname{Im} d_{n+1} = A_{n+1}$. It is easy to check that the h_n's define a contracting homotopy:

$$(h_{n-1}d_n + d_{n+1}h_n)(a+b) = h_{n-1}d_n(b) + d_{n+1}h_n(a) = s_n d_n(b) + d_{n+1}(s_{n+1}(a))$$

This gives the identity because b belongs to the image of s_n. □

In the next proposition we will use a tensor product of chain complexes. This has already appeared briefly in the exercises (and in other courses?), but let us recall that $(F_\bullet \otimes F'_\bullet)_n$ is the direct sum of all $F_i \otimes F'_j$ where $i+j=n$ and the differential D is given on this component by the formula $D(x \otimes x') = dx \otimes x' + (-1)^i x \otimes d'x'$. The mnemonic trick to remember the sign is that the differential d' has to cross a degree i element before moving to the other side of the tensor product, and each degree has (-1) cost.

Proposition 5.2. *Let $F_\bullet$ and $F'_\bullet$ be free resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$ respectively. Then $F_\bullet \otimes F'_\bullet$ is free resolution of $\mathbb{Z}$ over $\mathbb{Z}(G \times G')$.*

PROOF. The augmentations $\varepsilon : F_0 \to \mathbb{Z}$ and $\varepsilon' : F'_0 \to \mathbb{Z}$ define a new augmentation $(F \otimes F')_0 = F_0 \otimes F'_0 \to \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$. We have discussed this at the beginning, the tensor product is one of free $\mathbb{Z}(G \times G')$-modules, so we are done if we show that the coaugmented complex is acyclic.

This we check over $\mathbb{Z}$ by a double application of Lemma 5.1. Since $F_\bullet \xrightarrow{\varepsilon} \mathbb{Z}$ is an acyclic resolution by free abelian groups, it is contractible. This is equivalent to saying that $F_\bullet$ is homotopy equivalent to the chain complex $\mathbb{Z}$ concentrated in degree zero: the section s_0 provides a homotopy inverse to the augmentation map $\varepsilon \colon F_\bullet \to \mathbb{Z}$ where the contracting homotopy constructed in Lemma 5.1 is a homotopy between $s_0 \circ \varepsilon$ and the identity.

Thus $\mathbb{Z} \otimes F'_\bullet$ is homotopy equivalent to $F_\bullet \otimes F'_\bullet$. The former is $F'_\bullet$, which is homotopy equivalent to $\mathbb{Z}$ for the same reason. All together we have shown that $\varepsilon \otimes \varepsilon' \colon F_\bullet \otimes F'_\bullet \to \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$ is a homotopy equivalence over $\mathbb{Z}$, hence acyclic in the category of $\mathbb{Z}(G \times G')$-modules. □

Corollary 5.3. *Let $F_\bullet$ and $F'_\bullet$ be free resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$. Then so is $F_\bullet \otimes F'_\bullet$.*

PROOF. The previous proposition shows that the tensor product $F_\bullet \otimes F'_\bullet$ is a free $\mathbb{Z}(G \times G)$-resolution of $\mathbb{Z}$. We let G act on this resolution via the diagonal map. □

Proposition 5.4. *Let G, G' be two groups, M, M' be left $\mathbb{Z}G$-, respectively $\mathbb{Z}G'$-modules, and $F_\bullet$, $F'_\bullet$ be free resolutions of $\mathbb{Z}$ over the respective group rings. There is an isomorphism of chain complexes of $\mathbb{Z}$-modules*

$$(F_\bullet \otimes_{\mathbb{Z}G} M) \otimes (F'_\bullet \otimes_{\mathbb{Z}G'} M') \to (F_\bullet \otimes F'_\bullet) \otimes_{\mathbb{Z}(G\times G')} (M \otimes M')$$

Proof. On the left we have a quotient of $(F_\bullet \otimes M) \otimes (F'_\bullet \otimes M')$, taking coinvariants under the action of G and G' on the two first components and the two last ones separately. Given an element $x \otimes m \in F_i \otimes M$ and $x' \otimes m' \in F'_j \otimes M'$, we send it to the class of $(x \otimes x') \otimes (m \otimes m')$. This is compatible with the coinvariants we take on the left since $xg \otimes x'$ is in the orbit of $x \otimes x'$ under the action of $(g, 1) \in G \times G'$.

This small computation shows in fact that the induced map is an isomorphism degreewise. It is moreover a chain map as one can check directly. □

Let $z \in F_i \otimes_{\mathbb{Z}G} M$ and $z' \in F_j \otimes_{\mathbb{Z}G} M'$ be two elements. We denote by $z \times z'$ its image under the isomorphism in Proposition 5.4. We call $n = i + j$. The formula for the differential in a tensor product shows that $d(z \times z') = dz \times z' + (-1)^i z \times dz'$, hence the image of two cycles z, z' is a cycle.

Definition 5.5. Let $z \in F_i \otimes_{\mathbb{Z}G} M$ and $z' \in F_j \otimes_{\mathbb{Z}G} M'$ be two cycles representing classes $cl(z) \in H_i(G; M)$ and $cl(z') \in H_j(G'; M')$. The class of the cycle $z \times z'$ is the *cross product* of $cl(z)$ and $cl(z')$. It defines a class $cl(z) \times cl(z') \in H_n(G \times G'; M \otimes M')$.

This definition makes sense because $F_\bullet \otimes F'_\bullet$ is a free resolution of $\mathbb{Z}$ over the group ring $\mathbb{Z}(G \times G')$ by Proposition 5.2. It sends a boundary to a boundary – because for any cycle z' we have $du \times z' = d(u \times z')$ – hence is well defined on homology classes. It is finally independent of the choices of resolutions $F_\bullet$ and $F'_\bullet$ by standard homological algebra arguments: free (or projective) resolutions are unique up to canonical homotopy equivalence.

There is a completely analogous definition of a cross product in cohomology. Given cochains $u \in \mathrm{Hom}_{\mathbb{Z}G}(F_i, M)$ and $u' \in \mathrm{Hom}_{\mathbb{Z}G'}(F'_j, M')$, we define

$$u \otimes u' \in \mathrm{Hom}_{\mathbb{Z}(G\times G')}(F_i \otimes F'_j, M \otimes M')$$

as the tensor product, and extend to a cochain $u \times u' \colon F_\bullet \otimes F'_\bullet \to M \otimes M'$ of degree $i + j = n$ which is zero on $F_k \otimes F'_{n-k}$ for $k \neq i$.

Definition 5.6. The *cohomological cross product* of two classes $cl(u) \in H^i(G; M)$ and $cl(u') \in H^j(G'; M')$ is the class $cl(u) \times cl(u') \in H^{i+j}(G \times G'; M \otimes M')$ represented by $u \times u'$.

The coboundary of a cross product $d(u \times u')$ can be computed by evaluating on an element $x_{i+1} \otimes y_j \in F_{i+1} \otimes F_j$, and $x_i \otimes y_{j+1} \in F_i \otimes F_{j+1}$. We see that $d(u \times u') = du \times u' + (-1)^i u \times du'$. This shows in particular that the cross product of cocyles is a cocycle, and if $u = dU$ is a coboundary, then the cross product of dU with a cocycle u' is a coboundary as well since $dU \times u' = d(U \times u')$ (and likewise on the other side of the tensor product). Therefore this operation is well defined.

6. Functoriality of (co)homology

In the previous section we have made convenient choices of resolutions, they helped us to give simple maps at the chain and cochain level. In the exercises we have also encountered the effect on homology of the inclusion of a subgroup. We should probably take some time now to explain that group (co)homology is a functor.

Let $\alpha \colon G \to G'$ be a group homomorphism, M be a $\mathbb{Z}G$-module, M' be a $\mathbb{Z}G'$-module, and $f \colon M \to M'$ a homomorphism of $\mathbb{Z}$-modules such that $f(gm) = \alpha(g)f(m)$ for any $g \in G$ and $m \in M$. A typical map f could be the identity on a $\mathbb{Z}G'$-module M' where we see the source $M = \alpha^* M'$ as the abelian group M' with G-action given through α by $gm' = \alpha(g)m'$. We will mostly deal with this case. The identity on the trivial module $\mathbb{Z}$ is an example of such a map where we consider the copy on the right as a $\mathbb{Z}G'$-module and the one on the left as a $\mathbb{Z}G$-module by letting G act (trivially) via α.

Proposition 6.1. *There is a well-defined map* $(\alpha, f)_* \colon H_*(G; M) \to H_*(G'; M')$.

Proof. Let $F_\bullet$ be a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ and $F'_\bullet$ be a free $\mathbb{Z}G'$-resolution of $\mathbb{Z}$. We use a *left* action by the group on the module here (but remember that this can be reversed to a right action by the inverse). It extends to a G-equivariant chain map $\tau \colon F_\bullet \to F'_\bullet$ via α where we see the latter as an acyclic complex of $\mathbb{Z}G$-modules. The homomorphism α extends to a map of group rings $\alpha \colon \mathbb{Z}G \to \mathbb{Z}G'$, and then on the entire resolutions.

Anyway, τ and f induce a map $\tau \otimes f \colon F_\bullet \otimes_{\mathbb{Z}G} M \to F'_\bullet \otimes_{\mathbb{Z}G'} M'$. This is well defined since $\tau(gx) \otimes f(gm) = \alpha(g)\tau(x) \otimes \alpha(g)f(m)$ by our equivariant contruction of τ and the property we require of f. □

Remark 6.2. If we choose to work with the standard bar resolutions, then we see that the construction is functorial (and we do not have to construct a complicated chain map τ). The obvious choice for τ is to send a generator $[g_1|\dots|g_n]$ to $[\alpha(g_1)|\dots|\alpha(g_n)]$.

Example 6.3. Let $H < G$ be a subgroup and $g \in G$. We choose $\alpha = c_g \colon H \to gHg^{-1}$ to be conjugation by g. If M is a $\mathbb{Z}G$-module, then it is a $\mathbb{Z}H$-module and a $\mathbb{Z}(gHg^{-1})$-module by restriction. The map $f \colon M \to M$ given by left multiplication by g is compatible with the respective module structures over H and its conjugate:

$$f(h \cdot m) = g(hm) = c_g(h)gm = \alpha(h) \cdot f(m)$$

for any $h \in H$ and $g \in G$. To compute the effect of c_g in homology we can choose a free resolution $\mathbb{F}_\bullet$ of $\mathbb{Z}$ over $\mathbb{Z}G$, and we observe that it consists also of free $\mathbb{Z}H$-modules over any subgroup $H < G$.

Let us check that multiplication by g on $F_\bullet$ is a chain map we are looking for. It is a chain map since we took care in choosing a resolution over $\mathbb{Z}G$, and then, for any $x \in F_n$, the image of hx is $g(hx) = c_g(h)gx$, this is the same computation as above. Therefore the map induced on homology $H_*(H;M) \to H_*(gHg^{-1};M)$ is given at the chain level by the map $F_\bullet \otimes_{\mathbb{Z}H} M \to F_\bullet \otimes_{\mathbb{Z}(gHg^{-1})} M$ sending $x \otimes m$ to $gx \otimes gm$.

In this way conjugation by $g \in G$ yields a map written $g \cdot (-)$. In particular, when $g = h$ is an element in H, then the conjugated copy of H is H again and the element $hx \otimes hm$ lies in the same orbit as $x \otimes m$ so that conjugation by h induces the identity on $H_*(H;M)$.

In cohomology the situation is analogous. Any chain map $F_\bullet \to F'_\bullet$ as above induces a cochain map

$$\alpha^* \colon \operatorname{Hom}_{\mathbb{Z}G'}(F'_\bullet; M') \to \operatorname{Hom}_{\mathbb{Z}G}(F_\bullet; M)$$

The same argument as above shows that there is a conjugation homomorphism on the cohomology of $H < G$ and that of some conjugate of H, which is the identity when we conjugate by an element of H. Let us record these two facts in a proposition.

Proposition 6.4. *Let G be a group and $g \in G$ any element. Then, the conjugation map $c_g \colon G \to G$ induces the identity maps $(c_g)_* \colon H_*(G;M) \to H_*(G;M)$ and $(c_g)^* \colon H^*(G;M) \to H^*(G;M)$ for any $\mathbb{Z}G$-module M.* □

7. The cup product

We arrive now at the *internal* cup product in cohomology. The two key ingredients are the cohomological cross product defined in the previous section, and the diagonal map $\Delta \colon G \to G \times G$. There is no "codiagonal map", which explains why the cup product has no homological companion. In this section we use a single group and two $\mathbb{Z}G$-modules which we call M and N. The tensor product $M \otimes N$ is a $\mathbb{Z}(G \times G)$-module as in the previous section, and becomes a G-module equipped with the diagonal G-action.

Definition 7.1. Let $u \in H^i(G;M)$ and $v \in H^{n-i}(G;N)$. The *cup product* $u \cup v$ in $H^n(G; M \otimes N)$ is the image under $\Delta^* \colon H^n(G \times G; M \otimes N) \to H^n(G; M \otimes N)$ of the cross product $u \times v$.

At the level of cochains, the following takes place. Let $F_\bullet$ be a free $\mathbb{Z}G$-resolution of the trivial module $\mathbb{Z}$, for example the bar resolution. We have seen in Corollary 5.3 that $F_\bullet \otimes F_\bullet$ is again a free resolution of $\mathbb{Z}$, equipped with the diagonal action of G. The tensor product of cochains, or better said cocycles, provides a well-defined map $F_\bullet \otimes F_\bullet \to M \otimes N$. When seen as a morphism of $\mathbb{Z}(G \times G)$-modules this defines the cross product, and when considered only with the diagonal G-action, this defines the cup product.

Remark 7.2. Again, there seem to be choices involved in the definition of the cup product, but they do not matter by uniqueness of resolutions up to homotopy equivalence, see for example [**7**, Theorem 1.7.5]. Therefore, if one would prefer to use a smaller resolution of $\mathbb{Z}$ than a tensor product, one could choose a *diagonal approximation* $F_\bullet \to F_\bullet \otimes F_\bullet$, that is, a lift of the identity on $\mathbb{Z}$, thus necessarily unique up to homotopy by the fundamental theorem of homological algebra, to obtain a cocycle representative in our favorite resolution $F_\bullet$. We will see one below, for the bar resolution.

Remark 7.3. This information comes mainly from the Wikipedia page on Hassler Whitney (1907-1989). His name will appear in the next example because of the Alexander-Whitney map, but it should have appeared before, and the same applies to James Alexander since he was probably the first to dualize the chain complex so as to obtain a cochain complex (for homology of spaces). You might have heard about the Alexander polynomial in knot theory. In 1954 he signed a letter to support Oppenheimer and since then lived isolated during McCarthyism. As a younger man he loved climbing, even buildings in Princeton and would always leave his window open so that he could enter his apartment through the window.

It is in the Swiss Alps, where he was climbing with Georges de Rham, that he met Whitney, a talented alpinist as well. Whitney's ashes have been placed by UNIL maths professor Oscar Burlet on the top of la Dent Blanche in the summer of 1989. His PhD thesis was about graph theory, but he defined first what a smooth manifold of class C^r is in 1936, and introduced the cup product.

He was someone quite remarkable and broad, building a house still known today as the Whitney House in Massachusetts with an innovative architect, playing violin and viola, running between 6 to 12 miles every other day, plus mountaineering of course (he was member of the Swiss Alpine Club).

Example 7.4. Let $F_\bullet$ be the bar resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. There is another resolution given by $F_\bullet \otimes F_\bullet$. In degree n it consists in

$$\bigoplus_{i=0}^{n} F_i \otimes F_{n-i} = \bigoplus_{i=0}^{n} \mathbb{Z}[G^{i+1}] \otimes \mathbb{Z}[G^{n-i+1}]$$

There is a diagonal approximation $\Delta\colon F_\bullet \to F_\bullet \otimes F_\bullet$ classically known as the Alexander-Whitney map, given on generators $(g_0, \ldots, g_n) \in G^{n+1}$ by the formula

$$\Delta(g_0, \ldots, g_n) = \sum_{i=0}^{n} (g_0, \ldots, g_i) \otimes (g_i, \ldots, g_n)$$

This is clearly compatible with the diagonal G-action. It is the diagonal in degree zero $\mathbb{Z}G \to \mathbb{Z}G \otimes \mathbb{Z}G$ and forms a chain map as we can check directly, but let us only do the computation in degree 2:

$$\begin{aligned}\Delta(d(g,h,k)) &= \Delta((h,k)-(g,k)+(g,h))\\ &= (h)\otimes(h,k)+(h,k)\otimes(k)-(g)\otimes(g,k)-(g,k)\otimes(k)+(g)\otimes(g,h)+(g,h)\otimes(h)\\ &= (g)\otimes[(h,k)-(g,k)+(g,h)]+[(h)-(g)]\otimes(h,k)-(g,h)\otimes[(k)-(h)]+\\ &+[(h,k)-(g,k)+(g,h)]\otimes(k) = d((g)\otimes(g,h,k)+(g,h)\otimes(h,k)+(g,h,k)\otimes(k)\\ &= d(\Delta(g,h,k))\end{aligned}$$

This means that we can compute a cup product in the bar resolution, by pulling back the cross product through a diagonal approximation. We offer a small computation for the cyclic group of order 2.

Example 7.5. Let us use trivial coefficients in the field $\mathbb{F}_2$ of two elements. A cochain is a $\mathbb{Z}G$-module homomorphism $\mathbb{Z}[G^{n+1}] \to \mathbb{F}_2$, which is determined by its value on $[g_1|\dots|g_n]$ in the bar notation. We focus on the cyclic group C_2, so here, g_i is either 1 or t. The only non-trivial cochain in degree zero sends $[]$ to 1, it is a cocycle and represents an element $1 \in H^0(C_2;\mathbb{F}_2)$.

A 1-cochain is determined by its value on $[1]$ and $[t]$. But $[1] = d[1|1]$ is a boundary, so a cocycle must send it to 0. As we know that $H^1(C_2;\mathbb{F}_2) \cong \mathbb{F}_2$, there must be a non-trivial cocycle, and in fact it is not hard to check that sending $[t]$ to 1 defines a cocycle u, representing a cohomology class we also call u. Before trying to compute $u \cup u$ let us try to understand $H^2(C_2;\mathbb{F}_2)$.

We claim that the cochain defined by sending all generators to zero, except $[t|t]$ to 1 is a cocycle. Indeed the image under the differential in the bar resolution of $[g|h|k]$ is an alternating sum of elements that cancel out or contain a 1 in the bar notation (for example by checking the case where one element is 1, or none is so). It cannot be a coboundary because $d[t|t]$ and $d[1|1]$ differ by $t \cdot [t] + [t]$, an element a cochain must send to zero in a trivial module, so those two generators cannot have the same value.

The cup product $u \cup u$ is given by the composition of the Alexander-Whitney map, followed by the cochain $u \times u$. Notice that Δ sends $[g|h]$, which corresponds to the triple $(1, g, gh)$ to the sum $(1) \otimes (1, g, gh) + (1, g) \otimes (g, gh) + (1, g, gh) \otimes (gh)$. In the bar notation we get $[] \otimes [g|h] + [g] \otimes g[h] + [g|h] \otimes gh[]$. The cross product is zero on the first and last term (by definition we use $u \otimes u$ on the middle term and extend it to a map on $(F_\bullet \otimes F_\bullet)_2$ by sending $F_0 \otimes F_2$ and $F_2 \otimes F_0$ to zero), so we compute it on the middle one.

Whenever g or h equals 1, we find zero since $u[1] = 0$, but on $[t|t]$ we obtain $u[t] \times u[t] = 1$. This represents the only non-trivial cohomology class in degree 2, so $u \cup u \neq 0$.

However one can probably do better with a smaller resolution. Let us combine, in the case of the cyclic group of order 2, the two reduction steps, first the use of an explicit periodic resolution and second an ad hoc diagonal approximation.

Example 7.6. We consider the resolution with $F_n = \mathbb{Z}C_2$ for all n, one out of two maps is $t-1$, the others are $t+1$. The tensor product $F_\bullet \otimes F_\bullet$ has therefore in degree n a direct sum of $n+1$ copies of $\mathbb{Z}C_2 \otimes \mathbb{Z}C_2$.

We construct a diagonal approximation Δ from $F_\bullet$ by assigning to $1 \in \mathbb{Z}C_2 = F_n$ the sum of the following elements:

$$\begin{cases} 1 \otimes 1 \in F_{2i} \otimes F_j & \text{for } 2i + j = n \\ 1 \otimes t \in F_{2i+1} \otimes F_j & \text{for } 2i + j + 1 = n \end{cases}$$

In degree zero we have thus the diagonal as in the previous example, and higher degrees the image of 1 is a sum of $1 \otimes 1$'s and $1 \otimes t$'s. One can check that this is indeed a chain map, a good exercise is to do it by hand in degrees 0, 1, and 2 to see why this somewhat exotic formula works. To do the formal computation it would be best to do an induction and distinguish even and odd degrees.

Assuming this is fine, let us denote by u_n the unique non-trivial cohomology class in $H^n(C_2; \mathbb{F}_2) \cong \mathbb{F}_2$, represented in the periodic cochain complex by $u_n \colon \mathbb{Z}C_2 \to \mathbb{F}_2$ that sends 1 to 1 (and t to 1 as well). Remember that all differentials become zero when homing into $\mathbb{F}_2$ because signs to not matter anymore, we reduce mod 2, and the C_2-action is trivial on the coefficient module $\mathbb{F}_2$. To compute the cup product $u_i \cup u_j$ we have to pullback the cross product along the diagonal approximation and from our choice of cocycles we can focus on $F_i \otimes F_j$ as one extends this to all other terms in $(F_\bullet \otimes F_\bullet)_{i+j}$ by the zero map. This corresponds to the composition

$$\mathbb{Z}C_2 \to \mathbb{Z}C_2 \otimes \mathbb{Z}C_2 \xrightarrow{u_i \otimes u_j} \mathbb{F}_2 \otimes \mathbb{F}_2 \cong \mathbb{F}_2$$

This map sends 1 either to $1 \otimes 1$ or $1 \otimes t$, depending on the parity of i, but in the end both choices are sent to $1 \otimes 1$, i.e. to $1 \in \mathbb{F}_2$. Hence $u_i \cup u_j = u_{i+j}$. In particular u_0 is a unit for the cup product.

We have identified our first cohomology ring, as a graded $\mathbb{F}_2$-algebra.

Proposition 7.7. *There is an isomorphism $H^*(C_2; \mathbb{F}_2) \cong \mathbb{F}_2[u]$, where u is a generator in degree one.*

Proof. We already know that the map sending u_j to u^j is an isomorphism of graded vector spaces. The computation in the previous example shows that u_n is the iterated cup product of u with itself n times. The isomorphism respects thus the multiplicative structure. □

We will do the analogous computation for the cyclic group of order 3 in the exercises.

8. Properties of the cup product

Our aim is to prove that group cohomology enjoys all the expected property a graded commutative ring should have (say with trivial coefficients). We will also mention naturality properties in the group variable and the module. We fix a group G and will use $\mathbb{Z}G$-modules M, N, L.

Let us start by identifying the effect in degree 0.

Lemma 8.1. *The cup product $H^0(G; M) \otimes H^0(G; N) \to H^0(G; M \otimes N)$ is the map $M^G \otimes N^G \to (M \otimes N)^G$ induced by the inclusions $M^G \subset M$ and $N^G \subset N$.*

Proof. We know that $H^0(G; M) \cong M^G$, the zeroth cohomology group is given by the G-invariants. Given G-invariant elements $m \in M^G$ and $n \in N^G$, we represent them as cocycles, i.e., $\mathbb{Z}G$-module maps $\mathbb{Z}G \to M$ sending 1 to m (and respectively to n). Their cross product is then $m \otimes n$ and the cup product is obtained by precomposing with the diagonal the map

$$\mathbb{Z}G \xrightarrow{\Delta} \mathbb{Z}G \otimes \mathbb{Z}G \xrightarrow{m \otimes n} M \otimes N.$$

This cochain is the cocycle sending 1 to $m \otimes n$, a G-invariant element in the tensor product. □

Lemma 8.2. *The element $1 \in H^0(G;\mathbb{Z}) = \mathbb{Z}$ is the unit for the cup product.*

Proof. Let $u \in H^n(G;M)$ be a cohomology class represented by a cocycle $u\colon F_n \to M$. Notice that our cohomology class 1 is represented by the cocycle $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$, the coaugmentation map (compare with the previous proof). The cross product $u \times 1$ is then given by a cocycle defined on $F_n \otimes F_0 \subset (F_\bullet \otimes F_\bullet)_n$. To detect what this class is in $H^n(G; M \otimes \mathbb{Z}) \cong H^n(G;M)$, we use the morphism of chain complexes $\mathrm{id} \otimes \varepsilon\colon F_\bullet \otimes F_\bullet \to F_\bullet \otimes \mathbb{Z}$. Both complexes are resolutions of M by free $\mathbb{Z}G$-modules, this chain map therefore induces the identity in cohomology. Obviously $u \cup 1$ corresponds to u in the standard bar resolution. The same argument applies to prove that $1 \cup u = u$. □

Proposition 8.3. *The cup product is associative: $u \cup (v \cup w) = (u \cup v) \cup w$ in $H^*(G; L \otimes M \otimes N)$.*

Proof. Associativity already holds at the cochain level for the cross product. □

Example 8.4. We compute the cohomology $H^*(\mathbb{Z} \times \mathbb{Z}; \mathbb{Z})$ with the multiplicative structure. To distinguish the trivial coefficients from the group, we write the latter in multiplicative notation $C = \langle t \rangle$. The augmented chain complex $\mathbb{Z}C \xrightarrow{t-1} \mathbb{Z}C \xrightarrow{\varepsilon} \mathbb{Z}$ is a free $\mathbb{Z}C$-resolution of $\mathbb{Z}$ as we have seen in Example 2.3. Therefore the tensor product with itself

$$0 \to \mathbb{Z}(C \times C) \to \mathbb{Z}(C \times C) \oplus \mathbb{Z}(C \times C) \to \mathbb{Z}(C \times C) \to \mathbb{Z}$$

is a free $\mathbb{Z}(C \times C)$-resolution of $\mathbb{Z}$. Hom-ing into $\mathbb{Z}$, and forgetting about the coaugmentation, we get a small cochain complex $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ where both maps are zero as they are induced by $t - 1$. We recover the graded $\mathbb{Z}$-module we already know (this is integral cohomology of the torus $S^1 \times S^1$). The integral cohomology of $C \times C$ consists in four copies of $\mathbb{Z}$, one in degree zero, two in degree 1, and the last one in degree 2.

The generator in degree 0 is the unit 1, and we call u_1 and u_2 the generators in degree 1. The infinite cyclic group C is a retract of $C \times C$ in two different ways: through the inclusion $i_k\colon C \hookrightarrow C \times C$ and the projection $\pi_k\colon C \times C \twoheadrightarrow C$ onto the

k-th factor for $k = 1, 2$. We can then identify the generators $u_k = (\pi_k)^*(u)$. In terms of the bar resolution this means for example that

$$u_1[t_1^m t_2^n] = u[t^m] = m$$

Recall that the boundary map in the bar resolution is $d_2[g|h] = g[h] - [gh] + [g]$ so that a cocyle with coefficients into a trivial module must send $[gh]$ to the same image as the sum of the images of $[g]$ and $[h]$. The above formula for u defines then indeed a non-trivial cocycle in $\mathrm{Hom}_{\mathbb{Z}C}(\mathbb{Z}[C^2], \mathbb{Z})$ since the images of $[t^m] + [t^n]$ and of $[t^{m+n}]$ agree. It is not a coboundary of the form $\mathbb{Z}[C^2] \to \mathbb{Z}C \xrightarrow{f} \mathbb{Z}$ because the image of $[g]$ here is $f(g[]) - f[] = 0$ as the $\mathbb{Z}C$-module structure on $\mathbb{Z}$ is trivial. The cocycle u cannot be divided by any integer, even up to a coboundary (look at the value on $[t]$) and it represents thus a generator of $H^1(C; \mathbb{Z}) \cong \mathbb{Z}$.

We can compute directly, or use the next result, that $u_1 \cup u_1 = 0 = u_2 \cup u_2$. The only potentially non-trivial cup product is therefore $u_1 \cup u_2$. To identify this class we use the Alexander-Whitney map from Example 7.4 and the bar resolution. We have to consider the composition

$$\mathbb{Z}[(C \times C)^3] \to \mathbb{Z}[(C \times C)^2] \otimes \mathbb{Z}[(C \times C)^2] \xrightarrow{u_1 \otimes u_2} \mathbb{Z}$$

where the first map is $(g, h, k) \mapsto (g, h) \otimes (h, k)$. In the bar notation it sends $[g|h] = (1, g, gh)$ to $(1, g) \otimes (g, gh) = [g] \otimes g[h]$, so all together the composite map representing the cup product sends

$$[t_1^m t_2^n | t_1^a t_2^b] \mapsto u_1(t_1^m t_2^n) \cdot t_1^m t_2^n u_2(t_1^a t_2^b) = mb.$$

This is a cocycle as can easily been verified (or trust me, if I did not do any mistake, then this must be a cocycle, by construction).

It is not a coboundary and cannot be divided as can be seen for example by noticing that for any cochain $\varphi\colon \mathbb{Z}[(C \times C)^2] \to \mathbb{Z}$, we have that $d\varphi$ has the same value on $[t_2|t_1]$ and $[t_1|t_2]$, namely $\varphi[t_1] + \varphi[t_2] - \varphi[t_1 t_2]$. But our cup product sends one to 1 and the other to zero.

All together we see that $H^*(C \times C; \mathbb{Z}) \cong E(u_1, u_2) = E(u_1) \otimes E(u_2)$, where $E(u)$ denotes $\mathbb{Z}[u]/(u^2)$. It consists in two copies of the integers, one generated by the unit 1 in dgeree zero, and one in degree 1, generated by u.

These first properties were quite formal and unsurprising, let us move to the first computation showing that commutativity in the graded sense is different to (ungraded) commutativity.

Proposition 8.5. *The cup product is graded commutative:* $u \cup v = (-1)^{pq} t_*(v \cup u)$ *in* $H^{p+q}(G; M \otimes N)$ *where* $t\colon N \otimes M \to M \otimes N$ *is the twist map sending* $n \otimes m$ *to* $m \otimes n$.

Proof. To compare the cup products $u \cup v$ and $v \cup u$ we introduce the chain map $\tau\colon F_\bullet \otimes F_\bullet \to F_\bullet \otimes F_\bullet$ defined on an element $x \otimes y \in F_p \otimes F_q$ by $\tau(x \otimes y) = (-1)^{pq}(y \otimes x)$. This is a chain map as one can check directly by computing $\tau[d(x \otimes y)]$. We obtain

$$\begin{aligned}\tau(dx \otimes y + (-1)^p x \otimes dy) &= (-1)^{(p-1)q} y \otimes dx + (-1)^{p+p(q-1)}(dy \otimes x) \\ &= (-1)^{pq}(dy \otimes x) + (-1)^{(p+1)q}(y \otimes dx)\end{aligned}$$

This is exactly $d[(-1)^{pq}(y \otimes x)] = d[\tau(x \otimes y)]$. We dualize now and consider the following commutative diagram

$$\begin{array}{ccc}\mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M) \otimes \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, N) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet \otimes F_\bullet, M \otimes N) \\ \downarrow{\scriptstyle \tau} & & {\scriptstyle \mathrm{Hom}(\tau, t)}\downarrow \\ \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, N) \otimes \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet \otimes F_\bullet, N \otimes M)\end{array}$$

We view on the right the tensor product of resolutions as a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ and notice that τ is compatible with the augmentation ε (because τ is the identity in degree zero). It induces therefore the identity in cohomology by uniqueness of resolutions up to homotopy. This shows that the map on the right induces t_* in cohomology so that the classes $t_*(u \otimes v)$ and $(-1)^{pq}(v \otimes u)$ agree in cohomology. □

We specialize now to the case $M = N = k$, an arbitrary field, seen as a trivial $\mathbb{Z}G$-module. Then $k \otimes k$ admits a map to k by multiplication which yields an internal cup product on $H^*(G; k)$. In degree zero $H^0(G; k) \cong k$, so the only degree zero classes are 1 and its scalar multiples.

Corollary 8.6. *The cohomology* $H^*(G; k)$ *is a graded commutative* k*-algebra.*

Proof. Since the cochain complex $\mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, k)$ is one of k-vector spaces, we know that the cohomology is a k-vector space in every degree. We have introduced a cup product, which is distributive with respect to the sum of cohomology classes since so is the cross product. The same is true for multiplication by scalars. The previous proposition tells us that commutativity has to be understood in the graded sense. □

Example 8.7. Let $\mathbb{F}_p$ be the field of p elements for some odd prime number p. Then any odd degree cohomology class u in $H^n(G; \mathbb{F}_p)$ must have square zero since $(-1)^{n^2}(u \cup u) = u \cup u$.

This shows in particular that the class $u \in H^1(C_p; \mathbb{F}_p)$ has square zero for any prime number $p \neq 2$, something we have computed explicitly in an exercise in the case of the cyclic group of order 3.

For the sake of completeness we state the obvious naturality property.

Proposition 8.8. *Let $\alpha\colon G \to H$ be a group homomorphism and N a $\mathbb{Z}H$-module. Then, for any $u, v \in H^*(H; N)$ we have $\alpha^*(u \cup v) = \alpha^* u \cup \alpha^* v \in H^*(G; N)$ where we see N as a $\mathbb{Z}G$-module through α.* □

We finally end this section and the chapter on group cohomology with the compatibility of the cup product with boundary homomorphisms. We apply this to understand better the cohomology of certain groups acting on spheres.

For now we fix a group and a short exact sequence of $\mathbb{Z}G$-modules

$$0 \to M' \to M \to M'' \to 0.$$

We also consider another $\mathbb{Z}G$-module N and assume that tensoring the above exact sequence with N yields another short exact sequence. Recall that the short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M') \to \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M) \to \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M'') \to 0$$

induces a long exact sequence in cohomology, which contains in particular a connecting homomorphism $\partial\colon H^p(G; M'') \to H^{p+1}(G; M')$.

Proposition 8.9. *For any $u \in H^p(G; M'')$ and $v \in H^q(G; N)$ we have $\partial(u \cup v) = \partial u \cup v$ in $H^{p+q+1}(G; M' \otimes N)$.*

Proof. The cup product with a representative cocycle v defines a homomorphism at the level of cochain complexes

$$C^\bullet(G, M) = \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M) \to C^{\bullet+q}(G; M \otimes N)$$

(defined with the bar resolution $F_\bullet$). This comes from the fact we have already met, that since v is a cocycle $dv = 0$, so that $d(u \cup v) = du \cup v$.

It raises the degree by q and doing the same for M' and M'' yields a commutative diagram of short exact sequences of cochain complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^\bullet(G, M') & \longrightarrow & C^\bullet(G, M) & \longrightarrow & C^\bullet(G, M'') & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle -\cup v} & & \downarrow{\scriptstyle -\cup v} & & \downarrow{\scriptstyle -\cup v} & & \| \\
0 & \longrightarrow & C^{\bullet+q}(G, M' \otimes N) & \longrightarrow & C^{\bullet+q}(G, M \otimes N) & \longrightarrow & C^{\bullet+q}(G, M'' \otimes N) & \longrightarrow & 0
\end{array}$$

We conclude by the naturality of connecting homomorphisms in long exact sequences.

□

Example 8.10. Let G be a finite group acting on a CW-complex X having the homotopy type of an odd dimensional sphere S^{2n-1} in such a way that the action permutes freely the cells. This is the case for example of a cyclic group C_n on S^1 or the quaternions Q_8 on S^3. We have seen in an exercise how the cellular chain complex $C_\bullet = C_\bullet^{cell}(X)$ can be spliced with itself iteratively so as to obtain a periodic free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. This is because the kernel of the last differential $d_{2n-1}\colon C_{2n-1} \to C_{2n-2}$ must be a copy of $\mathbb{Z}$ with the trivial G-action by a Lefschetz number argument.

The acyclic chain complex

$$0 \to \mathbb{Z} \to C_{2n-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

is not only acyclic, but contractible (as a chain complex of abelian groups). Now we remember that each C_i is a free $\mathbb{Z}G$-module and decompose then this exact sequence into $2n$ short exact sequences, starting with $\mathrm{Ker}\, d_0 \to C_0 \to \mathbb{Z}$, continuing with $\mathrm{Ker}\, d_1 \to C_1 \to \mathrm{Im}\, d_1 = \mathrm{Ker}\, d_0$, and ending with $\mathbb{Z} \to C_{2n-1} \to \mathrm{Ker}\, d_{2n-2}$.

The connecting homomorphisms associated to these short exact sequences are all isomorphisms in positive degrees as $H^k(G; \mathbb{Z}G) = 0$ for all $k > 0$. Therefore the composition of connecting homomorphisms ∂ given by

$$H^i(G;\mathbb{Z}) \cong H^{1+i}(G;\operatorname{Ker} d_0) \cong H^{2+i}(G;\operatorname{Ker} d_1) \cong \dots$$
$$\cong H^{2n-1+i}(G;\operatorname{Ker} d_{2n-2}) \cong H^{2n+i}(G;\mathbb{Z})$$

is also an isomorphism for any $i \geq 1$. When $i = 0$ all connecting maps are isomorphisms, except possibly the first one $H^0(G;\mathbb{Z}) \to H^1(G;\operatorname{Ker} d_0)$, which we know is a surjection as the next term in the cohomology long exact sequence is $H^1(G;C_0) = 0$.

Let us define u to be the image of $1 \in H^0(G;\mathbb{Z})$ in $H^{2n}(G;\mathbb{Z})$ and consider any element $v \in H^q(G;M)$. Then we have seen that the connecting homomorphisms are compatible with the cup product and if we write $v = 1 \cup v$ we obtain by Proposition 8.9 that the image under the iterated ∂'s is $d(1) \cup v = u \cup v$. Notice that since this map is an isomorphism in positive degree, this shows that u is non zero. The same argument works more generally for coefficients in a $\mathbb{Z}G$-module M, we will only use it for trivial modules such as $\mathbb{F}_p$, replacing the vanishing of cohomology with coefficients in $\mathbb{Z}G$ by coefficients in $\mathbb{F}_pG = \mathbb{Z}G \otimes \mathbb{F}_p$. In fact the statement you will prove in an exercise is about induced coefficients, which agree with coinduced coefficients when the group is assumed to be finite.

In the case of Q_8 this shows that the cup product with a generator u of the cyclic group $H^4(Q_8;\mathbb{Z})$ realizes an isomorphism $H^n(Q_8;\mathbb{Z}) \cong H^{n+4}(Q_8;\mathbb{Z})$. The multiplicative structure in degrees ≤ 3 therefore determines the whole graded ring structure.

To finish with a more elementary example, let us consider a cyclic group of odd prime order C_p. There is a class $u \in H^2(C_p;\mathbb{Z}) \cong \mathbb{Z}/p$ such that the isomorphism $H^1(C_p;\mathbb{F}_p) \cong H^{2n+1}(C_p;\mathbb{F}_p)$ is realized by $u^n \cup -$ and likewise for $H^0(C_p;\mathbb{F}_p) \cong H^{2n}(C_p;\mathbb{F}_p)$. Here the surjection from H^0 to H^2 is an isomorphism because we take mod p coefficients.

This determines completely the graded ring structure of $H^*(C_p;\mathbb{F}_p)$ together with the graded commutativity. If we call x a generator in degree one, we know that $x^2 = x \cup x = -x^2$, hence $x^2 = 0$. The above argument shows that the map

$$E(x) \otimes \mathbb{F}_p[u] \to H^*(C_p;\mathbb{F}_p)$$

that sends x to x and u to u is an isomorphism of $\mathbb{F}_p$-algebras.

CHAPTER 3

The Künneth formula

We start with a few recollections about the definition and elementary properties of singular/cellular cohomology. Our aim in this chapter is then to introduce the analog of the cross product we have encountered in group cohomology. For that it will be essential to understand the (co)homology of a product of two spaces, which will lead us to introduce the theory of so-called acyclic models.

1. A few recollections about singular cohomology

This part is classical and very few variations appear in different textbooks, Hatcher's book [**14**, Chapter 2] is a good reference, and it is probably familiar to everybody as it has been used in "Algebraic Topology". We present this for integral coefficients first and introduce quickly coefficients in arbitrary rings.

Let X be any topological space and $S_\bullet(X)$ (or maybe $C_\bullet^{sing}(X)$ sometimes) be the singular chain complex. Recall that $S_n(X)$ is the free abelian group on all maps $\Delta_n \to X$ whose source is the standard simplex Δ_n.

Definition 1.1. Let X ba a space. The *singular cochain complex* $S^\bullet(X)$ is the dual $\mathrm{Hom}(S_\bullet(X);\mathbb{Z})$ of the singular chain complex. The n-th *cohomology group* $H^n(X)$ is the n-th cohomology group of $S^\bullet(X)$.

For any ring R one defines cohomology with coefficients in R by considering $S^\bullet(X;R) = \mathrm{Hom}(S_\bullet(X);R)$ and defining cohomology with coefficients in R as $H^n(X;R) = H^n(S^\bullet(X;R))$. Likewise, there is a relative version yielding cohomology for pairs (X,A), where $A \subset X$ is a subspace of X.

Remark 1.2. For any non-empty space X there exists a map $\star \to X$ corresponding to the choice of a basepoint $x_0 \in X$. It provides a retract of the collapse map $X \to \star$, which in turn, by functoriality, shows that the cohomology of a point splits off $H^*(X;R)$.

Since ordinary cohomology of a point is concentrated in degree zero, $H^*(\star; R) \cong R$, this corresponds to the fact that the image of the unit $1 \in R \cong H^0(\star; R)$ is represented by the cochain $\varepsilon\colon S_0(X) \to R$ sending every 0-simplex $x \in X$ to 1. This element generates a copy of $R \subset H^0(X; R)$ that splits off.

This gives rise to reduced cohomology $\tilde{H}^n(X; R)$.

Just like homology, there is an axiomatic characterization of ordinary cohomology theory. It satisfies the well-known homotopy invariance property, excision and Mayer-Vietoris (and then long exact sequences for pairs), plus the dimension axiom asserting that cohomology of a point is concentrated in degree zero where it is isomorphic to a copy of the ring R.

Since cellular cohomology also satisfies these axioms, both theories coincide on CW-complexes and CW-pairs. Of course cellular homology does not make sense for arbitrary spaces so we cannot compare singular cohomology and cellular cohomology outside the category of CW-complexes. From now on we will however often omit to distinguish notationally cellular and singular cohomology for CW-complexes. When doing computations we will mainly use the cellular chain complex as it is more manageable.

2. The universal coefficient Theorem

We recall without proof the statement of the universal coefficient theorem, a useful theorem that allows one to compute cohomology groups from the homology groups, without redoing possibly similar computations which we have already done in homology. We illustrate and compare these two approaches in a series of examples.

Theorem 2.1. *Let (X, A) be a pair of spaces and n any natural number. Then there is a split short exact sequence*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H_{n-1}(X, A), R) \to H^n(X, A; R) \xrightarrow{\kappa} \mathrm{Hom}_{\mathbb{Z}}(H_n(X, A), R) \to 0$$

Proof. Let us just indicate how the map κ is defined. A cohomology class is represented by a cochain $c\colon S_n(X, A) \to R$, which is a cocycle i.e. $c \circ d_{n+1} = 0$. This shows that c actually defines a map of chain complexes $c\colon S_\bullet(X, A) \to R[n]$, where the target is the chain complex consisting of a single copy of R concentrated in degree n (with zero differentials). This map c induces therefore a homomorphism in homology $\kappa(c)\colon H_n(X, A) \to R$. $\square$

Example 2.2. Since the homology of a sphere S^n consists in free abelian groups (in degree 0 and n), the Ext-term vanishes and we obtain, for example, an isomorphism

$$H^n(S^n; R) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(S^n), R) \cong R$$

For the same reason $H^*(\mathbb{C}P^n; R)$ is concentrated in even degrees $0, 2, \ldots, 2n$ where $H^{2k}(\mathbb{C}P^n; R) \cong R$. In the presence of torsion, this is not the case anymore and Ext groups will appear. For $\mathbb{R}P^2$ for example we have that $H^k(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2$ for any $0 \leq k \leq n$.

3. Acyclic models

We introduce now the method of acyclic models. I have been toying with the idea to explain it only in the situation we need (to prove that the tensor product of singular chain complexes $S_\bullet(X)$ and $S_\bullet(Y)$ is a chain complex homotopy equivalent to $S_\bullet(X \times Y)$), but in the end my feeling is that the general method is not much more complicated, and maybe even easier to present. We follow quite closely the presentation in [**15**, Chapitre 8].

Definition 3.1. A *category with models* is a category $\mathcal{C}$ equipped with a set $\mathcal{M}$ of objects, called *models*.

This is of course too little to do anything serious, it is the choice of models that will allow us to say something about functors on $\mathcal{C}$.

Example 3.2. In the category Top of non-empty topological spaces one can choose all standard simplices Δ_n, for $n \geq 0$, as models. In the category Top^2 of pairs of non-empty topological spaces one can choose all pairs (Δ_n, Δ_m) of standard simplices.

Definition 3.3. Let $F\colon \mathcal{C} \to \mathrm{Ch}_\varepsilon(R)$ be a covariant functor from a category with models $\mathcal{M}$ to augmented graded chain complexes of R-modules. Then F is *acyclic relative to* $\mathcal{M}$ if $F(M)$ is acyclic (as an augmented chain complex) for any model $M \in \mathcal{M}$.

In other words the non-augmented chain complex determined by $F(M)$ is quasi-isomorphic to $R[0]$ via the augmentation ε.

Example 3.4. The singular chain functor $S_*(-;R)$ is acyclic relative to all standard simplices (they are all contractible). Observe that we need here the source category to consist of non-empty spaces if we want $S_*(X;R)$ to be augmented, as we use the map $X \to \star$ to induce $\varepsilon\colon S_0(X;R) \to S_0(\star;R) = R$. Likewise the functor sending a non-empty space X to $S_*(X \times I)$ is also acyclic relative to $\{\Delta_n \mid n \geq 0\}$.

An important functor is the functor defined on pairs of non-empty spaces by $(X,Y) \mapsto S_*(X \times Y;R)$ since we have to deal with this construction when computing the homology or cohomology of a product. It is acyclic relative to all pairs (Δ_n, Δ_m).

We arrive now at the key property a functor should have if we want to apply the method of acyclic models.

Definition 3.5. A functor $F\colon \mathcal{C} \to \mathrm{Ch}_\varepsilon(R)$ is *free relative to the models* $\mathcal{M}$ if for any $n \geq 0$ there is an index set J_n, models $M_\alpha \in \mathcal{M}$ for $\alpha \in J_n$, and a family of elements $x_\alpha \in F_n(M_\alpha)$ such that $F_n(X)$ is free as an R-module for any object $X \in \mathcal{C}$ on a basis given by $f_*(x_\alpha)$ where α runs over J_n and $f\colon M_\alpha \to X$ over all maps from M_α to X.

This definition is manufactured for the singular chain complex functor!

Example 3.6. The augmented singular chain complex is free relative to the standard simplices. Indeed, for $n \geq 0$ we choose the indexing set J_n to be a singleton $\{n\}$ and set $M_n = \Delta_n$. In $S_n(\Delta_n;R)$ we choose x_n to be the identity. Then it is clear that $S_n(X;R)$ is a free R-module over a basis given by all images of x_n under $f_*\colon S_n(\Delta_n;R) \to S_n(X;R)$.

The product functor on pairs of non-empty toopological spaces is also free, and again a singleton is enough for J_n. The model M_n is then the pair (Δ_n, Δ_n) and the element x_n is the diagonal map d_n. Since $S_n(X \times Y;R)$ is free over all simplices $\Delta_n \to X \times Y$ we see that they are images of d_n because such a simplex corersponds to a pair of singular simplices, one in X and the other one in Y. A map f here is precisely a pair of maps $f_X\colon \Delta_n \to X$ and $f_Y\colon \Delta_n \to Y$.

The cylinder functor $S_*(- \times I)$ is also free relative to all standard simplices, but the indexing set is larger now. We choose J_n to be the set of all maps $\Delta_n \to I$ and for each index α we set $M_\alpha = \Delta_n$. A singular simplex $\Delta_n \to X \times I$ is given by a pair of maps, namely a singular simplex in X and an index $\Delta_n \to I$. Therefore it suffices to choose x_α to be the singular simplex given by the identity and the index itself,

$$\Delta_n \xrightarrow{d_n} \Delta_n \times \Delta_n \xrightarrow{id \times \alpha} \Delta_n \times I.$$

In the following section we will use the previous examples in concrete situations so as to show that two functors are equivalent. The key result is this one. From now on we work with a category $\mathcal{C}$ having models $\mathcal{M}$ and two functors F and G from $\mathcal{C}$ to augmented chain complexes of R-modules.

Theorem 3.7. *Assume that F is free relative to $\mathcal{M}$ and that G is acyclic relative to $\mathcal{M}$. Then there exists a natural transformation $\tau\colon F \to G$ and moreover any two such natural transformations τ and τ' determine homotopic augmented chain maps $\tau_X, \tau'_X\colon F(X) \to G(X)$ through a natural homotopy.*

Proof. We construct τ inductively degree by degree. The induction starts by setting τ_X to be the identity on R in degree -1. We illustrate the induction step in degree zero. Let $J = J_0$ be the indexing set, and $x_\alpha \in F(M_\alpha)$ the basis given by the relative freeness of F, for $\alpha \in J$.

Since $G(M_\alpha)$ is acyclic the augmentation map $G_0(M_\alpha) \to R$ is surjective and we choose an element y_α such that $\varepsilon_Y(y_\alpha) = \varepsilon_X(x_\alpha)$. The assignment $x_\alpha \to y_\alpha$ is not unique, but provides a set of elements $y_\alpha \in G_0(M_\alpha)$.

We must now define τ_X in degree zero for all objects X. Since $F_0(X)$ is free on $f_*(x_\alpha)$ it is sufficient to define τ_X on this basis. Of course we choose $f_*(y_\alpha)$ and extend linearly.

By construction this map is compatible with the augmentations (it is so on the x_α's). Let us check that it defines a natural transformation in degree zero. Let $g\colon X \to Y$ be any morphism in $\mathcal{C}$, $f\colon M_\alpha \to X$ be any morphism from a model M_α

to X, for an index $\alpha \in J$, and consider the diagram

$$\begin{array}{ccccc} F_0(M_\alpha) & \xrightarrow{f_*} & F_0(X) & \xrightarrow{g_*} & F_0(Y) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau_X} & & \downarrow{\scriptstyle \tau_Y} \\ G_0(M_\alpha) & \xrightarrow{f_*} & G_0(X) & \xrightarrow{g_*} & G_0(Y) \end{array}$$

By freeness of $F_0(X)$ the right hand side square commutes if and only so does the outside rectangle, for all maps f, when evaluated on x_α. But this is the case by construction.

The general induction step is similar, we just need to replace the augmentation morphism by the differential. Let us assume that τ has been defined on all objects and all degrees up to $n-1$ in a natural way. With the same notation as above but $J = J_n$ this time, we look at the image of x_α under the differential $d_n \colon F_n(M_\alpha) \to F_{n-1}(M_\alpha)$. The element $\tau_X(d_n x_\alpha)$ is a cycle in $G_{n-1}(X)$ by naturality in low degrees, but by acyclicity it must be a boundary. We choose then an element $y_\alpha \in G_n(X)$ so that $d_n(y_\alpha) = \tau_X(d_n x_\alpha)$ and repeat the above argument mutatis mutandis.

To prove the second part of the theorem we consider two natural transformations (given maybe by different choices y_α in the suggested constructions described above). We must show that they are homotopic and for this we define inductively a homotopy h, which consists in a sequence of degree one maps $h_n \colon F_n(X) \to G_{n+1}(X)$. We set $h_{-1} = 0$.

Let $J = J_0$ and consider for any $x_\alpha \in F_0(M_\alpha)$ the difference $\tau(x_\alpha) - \tau'(x_\alpha)$ in $G_0(M_\alpha)$. By acyclicity this element, which is a cycle in the augmented chain complex $G_0(M_\alpha)$, must be a boundary, say of $y_\alpha \in G_1(M_\alpha)$. We decide then to define $h_0(x_\alpha) = y_\alpha$ and more generally, $h_0(X) \colon F_0(X) \to G_1(X)$ sends an element of the basis $f_*(x_\alpha)$ to $f_*(y_\alpha)$ and we extend linearly. It is easy to check that the difference $\tau_X - \tau'_X$ coincides with $d_1 \circ h_1$, by construction.

The induction step is similar but now h_{n-1}, which has been defined already, is not zero in general. Instead of the difference as above we have to consider the element

$$\tau(x_\alpha) - \tau'(x_\alpha) - h_{n-1}(d_n x_\alpha) \in G_n(M_\alpha)$$

When computing the boundary of this element, we have

$$d_n[\tau(x_\alpha)] - d_n[\tau'(x_\alpha)] - d_n[h_{n-1}(d_n x_\alpha)] = \tau(d_n x_\alpha) - \tau'(d_n x_\alpha) - d_n h_{n-1}(d_n x_\alpha)$$

But the homotopy h has been defined in degrees $\leq n-1$ in such a way that this is precisely $h_{n-2}d_{n-1}(d_n x_\alpha)$. This is zero because $d^2 = 0$, so again, we have a cycle, hence a boundary $d_{n+1}y_\alpha$. The same strategy as above works now without any problem to define h_n by sending x_α to $y_\alpha \in G_{n+1}(M_\alpha)$.

We have chosen these elements so that $dh + hd = \tau - \tau'$, naturality of h follows in the exact same way as what we have done for τ: Let $g\colon X \to Y$ be any morphism in $\mathcal{C}$, $f\colon M_\alpha \to X$ be any morphism from a model M_α to X, for an index $\alpha \in J$, and consider the diagram

$$\begin{array}{ccccc} x_\alpha \in F_n(M_\alpha) & \xrightarrow{f_*} & F_n(X) & \xrightarrow{g_*} & F_n(Y) \\ \downarrow{\scriptstyle h_\alpha} & & \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_Y} \\ y_\alpha \in G_{n+1}(M_\alpha) & \xrightarrow{f_*} & G_{n+1}(X) & \xrightarrow{g_*} & G_{n+1}(Y) \end{array}$$

By freeness of $F_n(X)$ the right hand side square commutes if and only so does the outside rectangle, for all maps f, when evaluated on x_α. But this is the case by construction and functoriality of F and G (by which we mean that $g_* \circ f_* = (g \circ f)_*$). This concludes the proof. $\square$

The following consequence is an exercise (Week 8).

Corollary 3.8. *Let $F, G\colon \mathcal{C} \to Ch_\varepsilon(R)$ be two functors that are acyclic and free relative to the models $\mathcal{M}$. The natural transformations $\tau\colon F \to G$ and $\sigma\colon G \to F$ provided by the acyclic models Theorem consist then of homotopy equivalences τ_X and σ_X for any $X \in \mathcal{C}$.*

Remark 3.9. The method of acyclic models is due to Eilenberg and Mac Lane in their 1953 paper [**10**]. The review on MathSciNet is written by Cartan himself. He observes that these techniques allow the authors to obtain easily previously known results (but without the cumbersome technical and computational aspects they used to involve). An extra benefit is that one can compare different versions of singular homology without any effort. He mentions that this article is the first one where cubical and simplicial homology are proven to be isomorphic.

Later other variants of this method have appeared, let us mention a more categorical one by Barr and Beck, [**4**], and also one by former EPFL algebra professor Michel André, [**3**].

4. The algebraic Künneth Theorem

In the next section we will come back to acyclic models and apply it to understand the homology of a product of spaces. In this section we treat first the purely algebraic problem of understanding the homology of a tensor product of chain complexes. We fix a PID R (so submodules of free modules are free, and all higher Tor_n^R vanish if $n \geq 2$) and two chain complexes $C_\bullet$ and $D_\bullet$ of R-modules. We start with the construction of a map from the tensor product of their homology groups to the homology of the tensor product.

Let $n = p + q$, $x \in Z_p(C_\bullet)$ be a degree p cycle, and $y \in Z_q(D_\bullet)$ be a degree q cycle. Then $x \otimes y \in C_p \otimes D_q$ is a degree n cycle in the tensor product $C_\bullet \otimes_R D_\bullet$ by definition of the differential. This allows us to define a map

$$\alpha\colon H_p(C_\bullet) \times H_q(D_\bullet) \to H_n(C_\bullet \otimes_R D_\bullet)$$

This map is clearly bilinear and well defined since for example, if $x = dx'$ is a boundary, then the image of (dx', y) is the boundary of $x' \otimes y$ (remember that y is a cycle), and likewise the image of an element of the form (x, dy') is the boundary of $x \otimes y'$, up to a $(-1)^p$ sign.

Lemma 4.1. *The maps α induce a homomorphism of R-modules*

$$\lambda\colon \bigoplus_{p+q=n} H_p(C_\bullet) \otimes_R H_q(D_\bullet) \to H_n(C_\bullet \otimes_R D_\bullet)$$

PROOF. This follows formally from the bilinearity of the maps α defined above. □

It is in general a difficult problem to determine the homology of a tensor product and we will see that even for chain complexes of modules over a PID and even if one of the two chain complexes is made of free modules, the map λ is not an isomorphism.

Lemma 4.2. *If $C_\bullet$ is a chain complex of free R-modules with zero differentials, then λ is an isomorphism.*

PROOF. The differential on $C_\bullet$ being trivial implies that the tensor product of chain complexes $C_\bullet \otimes_R D_\bullet$ splits as a direct sum $\oplus_p (C_p \otimes_R D_\bullet)$, the differential on $x \otimes y \in C_p \otimes_R D_q$ is simply $(-1)^p x \otimes dy$.

Therefore its homology splits as a direct sum of $H_*(C_p \otimes D_\bullet)$. For each integer p, we are assuming that C_p is a free R-module $C_p = \oplus_b R$ where b runs over a chosen basis. Hence $C_p \otimes_R D_\bullet \cong \oplus_b D_\bullet[p]$, and again the homology splits a direct sum

$$H_n(C_p \otimes_R D_\bullet) \cong \oplus_b H_{n-p}(D_\bullet) \cong C_p \otimes_R H_q(D_\bullet)$$

This concludes the proof. □

We are ready to state and prove the algebraic version of the Künneth Theorem.

Theorem 4.3. *Let R be a PID, $C_\bullet$ be a chain complex of free R-modules, and $D_\bullet$ a chain complex of R-modules. There is a short exact sequence for any $n \geq 0$:*

$$0 \to \bigoplus_{p+q=n} H_p(C_\bullet) \otimes_R H_q(D_\bullet) \xrightarrow{\lambda} H_n(C_\bullet \otimes_R D_\bullet) \xrightarrow{\mu} \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H_i(C_\bullet), H_j(D_\bullet) \to 0$$

PROOF. We decompose the chain complex $C_\bullet$ of free modules into two pieces that fit into the setting of the previous lemma. Let $Z_\bullet$ be the subcomplex of cycles $Z_n = Z_n(C_\bullet) \subset C_n$, which has zero differential. The quotient complex is then easily identified with $C_n/Z_n \cong B_{n-1}$, the shifted complex of boundaries $B_\bullet[1]$ where $B_n = \operatorname{Im} d_n \subset C_{n-1}$. Since $d^2 = 0$, the differential on $B_\bullet$ is also zero. We observe that both $Z_\bullet$ and $B_\bullet$ are chain complexes of free R-modules as they are subcomplexes of $C_\bullet$ and R is a PID.

Let us thus consider the short exact sequence of chain complexes

$$0 \to Z_\bullet \to C_\bullet \to B_\bullet[1]$$

Our next step is to take the tensor product with the chain complex $D_\bullet$, but we first realize that since B_m is a free R-module, all $\mathrm{Tor}_i^R(B_m, D_q)$ vanish for $i > 0$. We have thus short exact sequences

$$0 \to Z_p \otimes_R D_q \to C_p \otimes_R D_q \to B_{p-1} \otimes_R D_q \to 0$$

Taking direct sums over all pairs $p + q = n$ we get a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes_R D_\bullet \to C_\bullet \otimes_R D_\bullet \to B_\bullet[1] \otimes_R D_\bullet \to 0$$

Let us note that the differential on the left and the right is only induced by the differential on $D_\bullet$ (the one on cycles and boundaries is zero), but in the middle we have the usual differential on a tensor product of chain complexes. The exactness does not depend on the form of the differential, only on the horizontal arrows in these exact sequences.

To a short exact sequence of complexes we associate the long exact sequence in homology:

$$\cdots \to H_{n+1}(B_\bullet[1] \otimes_R D_\bullet) \xrightarrow{\partial_n} H_n(Z_\bullet \otimes_R D_\bullet) \to H_n(C_\bullet \otimes_R D_\bullet) \to H_n(B_\bullet[1] \otimes_R D_\bullet) \to$$

By Lemma 4.2 we can identify the homology groups of the source and target of the connecting homomorphism as

$$\partial_n \colon \bigoplus_{p+q=n} B_p \otimes_R H_q(D_\bullet) \to \bigoplus_{p+q=n} Z_p \otimes_R H_q(D_\bullet)$$

It is a standard fact that this homomorphism is induced by the inclusion $\iota_q \colon B_q \subset Z_q$ (this follows directly from the definition of ∂ by using the Snake Lemma). Therefore we see that $H_n(C_\bullet \otimes_R D_\bullet)$ is an extension of $\operatorname{Ker} \partial_{n-1}$ by $\operatorname{Coker} \partial_n$. The cokernel is easy to identify by right exactness of the tensor product and because $H_p(C_\bullet) = Z_p/B_p$. We obtain a direct sum of $H_p(C_\bullet) \otimes_R H_q(D_\bullet)$ as desired and the map to the homology of $C_\bullet \otimes_R D_\bullet$ is precisely λ.

To conclude we have to identify the kernel. The short exact sequence

$$0 \to B_{p-1} \to Z_{p-1} \to H_{p-1}(C_\bullet) \to 0$$

we have just used above is a resolution of the homology group by free R-modules. The $\operatorname{Tor} - \otimes$ exact sequence thus simplifies since all torsion products vanish on free modules and yields the expected description of the kernel as

$$\bigoplus_{p+q=n-1} \operatorname{Tor}^R(H_{p-1}(C_\bullet), H_q(D_\bullet))$$

This concludes the proof. □

Remark 4.4. The short exact sequence in the algebraic Künneth Theorem is natural in both variables. It actually also splits, but non naturally.

Hermann Künneth (1892-1975) was a German mathematician whose name is associated to many results about the homology or cohomology of a product, but he has had an uncommon career for such a famous name. His studies in mathematics in Erlangen and Munich were interrupted by the war. He served in the infantry, was wounded twice, was captured by the British forces and came back only in 1919. He finsihed his studies and became a high school teacher, but stayed in touch with the university. He wrote a PhD thesis in 1922 under the supervision of Tietze on the Betti numbers of a product of manifolds, including a result on the torsion part. He retired from high school in 1957 and was then appointed "ausserplanmässiger Professor" in Erlangen, where he continued doing research! (source: Wikipedia)

5. The topological Künneth Theorem

This section is devoted to the proof of the Künneth Theorem for topological spaces X and Y. The key ingredient to reduce the proof to that of the algebraic Künneth Theorem is the acyclic models Theorem as it will allow us to identify the singular chain complex of $X \times Y$ with the tensor product of the singular chain complexes for X and Y. We work with coefficients in an arbitrary ring, but the final steps will require this ring to be a PID.

Let $F = S_\bullet(- \times -; R)$ be the functor defined on a pair (X, Y) of non-empty spaces by the singular chain complex $S_\bullet(X \times Y; R)$ and let $G(X, Y)$ be the tensor product $S_\bullet(X; R) \otimes_R S_\bullet(Y; R)$.

Lemma 5.1. *The functor F is free and acyclic relative to the models $\{\Delta_n \mid n \geq 0\}$ and G is so relative to the models $\{(\Delta_m, \Delta_n) \mid m, n \geq 0\}$.*

Proof. The case of F has been discussed in Section 3, its freeness is the content of Example 3.6. As for G, we know that both $S_\bullet(\Delta_m; R)$ and $S_\bullet(\Delta_n; R)$ are acyclic augmented chain complexes, so that their tensor product is also acyclic by the algebraic Künneth Theorem 4.3.

We finally establish the freeness of G in any degree n by choosing the index set $J_n = \{(p, q) \in \mathbb{N} \times \mathbb{N} \mid p + q = n\}$. For each index (p, q) the chosen model is (Δ_p, Δ_q), and the element $x_{p,q} \in S_p(\Delta_p; R) \otimes_R S_q(\Delta_q; R)$ is given by the tensor product of the identities $\iota_p \otimes \iota_q$. Then their images under $f_* \otimes g_*$ for all pairs of singular simplices $f \colon \Delta_p \to X$ and $g \colon \Delta_q \to Y$ form a basis of the free R-module $S_p(X; R) \otimes_R S_q(Y; R)$.

The conclusion follows from the fact that G_n is a direct sum of all $S_p \otimes_R S_q$'s for $p+q=n$. $\square$

We are now ready for the Eilenberg-Zilber Theorem.

Theorem 5.2. *Let X and Y be non-empty spaces. The augmented chain complexes $S_\bullet(X \times Y; R)$ and $S_\bullet(X; R) \otimes_R S_\bullet(Y; R)$ are then naturally homotopy equivalent.*

Proof. The previous lemma allows us to apply Theorem 3.7 and its corollary. There exists a natural transformation $AW\colon S_\bullet(X \times Y; R) \to S_\bullet(X; R) \otimes_R S_\bullet(Y; R)$ and another one, $EZ\colon S_\bullet(X; R) \otimes_R S_\bullet(Y; R) \to S_\bullet(X \times Y; R)$, going the other way around. Both compositions are homotopic to the identity by uniqueness of such natural transformations (up to homotopy). $\square$

Remark 5.3. The way we obtained the *Alexander-Whitney map AW* is non-explicit. We only know that it exists, but just like we have seen with the bar construction in group cohomology, Example 7.4, there are explicit models, and often one refers to those as Alexander-Whitney maps. The same comment applies to the *Eilenberg-Zilber map EZ*. In this course we will not need an explicit description, which could be useful for other purposes.

We content ourselves with noting that any two AW maps are homotopic by the acyclic models Theorem, and likewise for EZ maps. Moreover two such maps are homotopy equivalences that are homotopy inverse to each other.

The topological Künneth Theorem is now a straightforward consequence of the algebraic one. We treat the absolute case.

Theorem 5.4. *Let X and Y be non-empty spaces and R a PID. There is a (natural) short exact sequence for any $n \geq 0$:*

$$0 \to \bigoplus_{p+q=n} H_p(X;R) \otimes_R H_q(Y;R) \to H_n(X \times Y; R) \xrightarrow{\mu} \bigoplus_{i+j=n-1} \mathrm{Tor}_R(H_i(X;R), H_j(Y;R)) \to 0$$

Proof. The homology of the product $X \times Y$ with coefficients in R is computed by the singular chain complex $S_\bullet(X \times Y; R)$. We have seen in Theorem 5.2 that this augmented chain complex is homotopy equivalent, via an Alexander-Whitney map, to the tensor product $S_\bullet(X;R) \otimes_R S_\bullet(Y;R)$. Its homology fits as stated in a short exact sequence when R is a PID as proven in the algebraic Künneth Theorem 4.3.

Notice that we refrained from calling the first map in the short exact sequence λ. It is in fact a composition of λ, the algebraic map with codomain the homology of the tensor product $S_\bullet(X;R) \otimes_R S_\bullet(Y;R)$, followed by an Eilenberg-Zilber map. $\square$

Remark 5.5. There is also a relative version of the Künneth Theorem which can be deduced from the absolute one, just like the relative version of the Universal Coefficients Theorem follows from the absolute one. Be careful however that it does only hold under some excisiveness assumptions, see [**6**]. Sometimes the relative case is stated without assumptions (and no proof), some other authors only care about CW-pairs, in which case the formula holds true.

The assumption one needs on pairs of subspaces (X, A) and (Y, B) (and we will see a counter example in the exercise sheet this week when it does not hold true) is that $\{X \times B, A \times Y\}$ be *excisive*. This means the following. The subcomplex $S_\bullet(X \times B \cup A \times Y)$ of $S_\bullet(X \times Y)$ contains the sum $S_\bullet(X \times B) + S_\bullet(A \times Y)$ as a subcomplex. We assume that this inclusion is a chain equivalence. This implies that the computation of the relative homology of the product pair $(X \times Y, X \times B \cup A \times Y)$ can be done instead by using $S_\bullet(X \times Y)/S_\bullet(X \times B) + S_\bullet(A \times Y) \cong S_\bullet(X, A) \otimes S_\bullet(Y, B)$.

Remark 5.6. Joseph A. Zilber was an American mathematician (1923-2009), studying and doing research at Harvard. He is best known for his work with Eilenberg, introducing simplicial sets and semi-simplicial sets in [**11**], and of course the topic of the present section, namely the study of the homology of a product via the EZ map, see [**12**]. He obtained his PhD in ... 1963 under the supervision of Gleason and Bott. He must have moved afterwards to more administrative jobs, like editing the mathematical reviews from the AMS for some years.

6. The cross product

There is a last thing we will do in homology before moving in the next chapter to singular cohomology: the cross product. There is probably no surprise in this last section as we have seen very similar statements in group homology.

We fix an Eilenberg-Zilber map $EZ\colon S_\bullet(X;R)\otimes_R S_\bullet(Y;R)\to S_\bullet(X\times Y;R)$ and an Alexander-Whitney map $AW\colon S_\bullet(X\times Y;R)\to S_\bullet(X;R)\otimes_R S_\bullet(Y;R)$, going the other way around. We stress the point that they are natural in X and Y by the acyclic model Theorem 3.7. We state the following lemma for the projection onto one variable, but the symmetric case for the other projection works equally well. It says that the topological projection onto one factor corresponds to the projection on one factor of the tensor product of singular chain complexes.

Lemma 6.1. *The following diagram is commutative up to homotopy:*

$$\begin{array}{ccc} S_\bullet(X;R)\otimes_R S_\bullet(Y;R) & \xrightarrow{EZ} & S_\bullet(X\times Y;R) \\ \downarrow{\scriptstyle id\otimes\varepsilon} & & \downarrow{\scriptstyle (p_1)_*} \\ S_\bullet(X;R)\otimes_R R & \xrightarrow{\cong} & S_\bullet(X;R) \end{array}$$

Proof. This diagram corresponds to the morphism in the category of pairs of non-empty topological spaces given by the identity on X and the projection $Y\to\star$, with the small difference that we should have $S_\bullet(X;R)\otimes_R S_\bullet(\star;R)$ in the lower left corner. The statement follows from the fact that the augmentation $S_\bullet(\star;R)\xrightarrow{\simeq}R$ is a chain equivalence of augmented chain complexes. □

We define next the cross product in homology and study its basic properties.

Definition 6.2. Let $p+q=n$. The *homological cross product* is the composition

$$H_p(X;R)\otimes_R H_q(Y;R)\xrightarrow{\lambda} H_n(S_\bullet(X;R)\otimes_R S_\bullet(Y;R))\xrightarrow{EZ} H_n(X\times Y;R)$$

We write $x\times y$ for the cross product of $x\in H_p(X;R)$ and $y\in H_q(Y;R)$.

It is immediate from the naturality of the map λ (by a small abuse we write λ here for its restriction to a single summand), and that of the Eilenberg-Zilber map, that the cross product is natural. We deduce next from Lemma 6.1 that the projection of the cross product on one component is zero most of the time (except when we compute the cross product with an element in $H_0(Y;R)$. Again we only state the case of the first projection $p_1\colon X\times Y\to X$. Let $\varepsilon\colon H_0(Y;R)\to R$ denote the map induced by $Y\to\star$ in homology, in other words this is the map induced in homology by the natural augmentation $\varepsilon\colon S_0(Y;R)\to R$ at the level of the singular chain complex.

Lemma 6.3. *Let* $p_1\colon X \times Y \to X$ *be the projection onto the first factor and let* $x \in H_p(X;R)$, $y \in H_q(Y;R)$. *Then* $(p_1)_*(x \times y) = 0$ *when* $q > 0$ *and if* $q = 0$

$$(p_1)_*(x \times y) = \varepsilon(y)x$$

Proof. The triviality in non-zero degrees follows from Lemma 6.1 since $(p_1)_*(y)$ lives in $H_q(\star;R)$, which is the trivial group. The second statement is also a consequence of the commutativity of the diagram in this lemma. □

The following expected properties of the cross product will be useful in the next chapter to show that the cup product equips the cohomology of a space with an R-algebra structure, which is commutative in the graded sense.

Proposition 6.4. *The cross product in homology is associative and commutative in the graded sense, i.e.,* $T_*(x \times y) = (-1)^{pq}(y \times x)$ *for* $x \in H_p(X;R)$, $y \in H_q(Y;R)$, *and* $T\colon X \times Y \to Y \times X$ *the map* $(x,y) \mapsto (y,x)$.

Proof. We consider the following diagram (where we drop the coefficient ring R from the notation):

$$\begin{array}{ccc} S_\bullet(X) \otimes S_\bullet(Y) \otimes S_\bullet(Z) & \xrightarrow{Id \otimes EZ} & S_\bullet(X) \otimes S_\bullet(Y \times Z) \\ \downarrow{\scriptstyle EZ \otimes Id} & & \downarrow{\scriptstyle EZ} \\ S_\bullet(X \times Y) \otimes S_\bullet(Z) & \xrightarrow{EZ} & S_\bullet(X \times Y \times Z) \end{array}$$

Both compositions provide a natural transformation from a free functor to an acyclic one, defined on triples of non-empty spaces where the models we choose are triples of standard simplices. We conclude from Theorem 3.7 that they must be homotopic. The triple cross product $(x \times y) \times z$ is the (homology class of the) image of the triple tensor product $x \otimes y \otimes z$ under the "left-bottom" composition, while $x \times (y \times z)$ is its image under the "top-right" composition. Homotopic chain maps induce the same homomorphism in homology, which proves associativity.

Graded commutativity follows from a similar argument to the one we have seen for the cup product in group cohomology, see Proposition 8.5.

Let $\tau\colon S_p(X) \otimes S_q(Y) \to S_q(Y) \otimes S_p(X)$ be the map $\tau(x \otimes y) = (-1)^{pq}(y \otimes x)$. This defines a morphism of chain complexes. Both $EZ \circ \tau$ and $T_* \circ EZ$ provide natural transformations $S_\bullet(X) \otimes S_\bullet(Y) \to S_\bullet(Y \times X)$, which must be homotopic by the acyclic models Theorem 3.7, hence induce the same map in homology. We compute now

$$\begin{aligned} T_*(x \times y) &= (T_* \circ EZ \circ \lambda)(x \otimes y) \\ &= (EZ \circ \tau_* \circ \lambda)(x \otimes y) \\ &= (EZ \circ \lambda)[(-1)^{pq}(y \otimes x)] \\ &= (-1)^{pq}(y \times x) \end{aligned}$$

where the first equality is the definition of the cross product, the second follows from the previous observation, the third one is the compatibility of λ with the chain map τ, and the fourth and last one is again the definition of the cross product. □

We end this chapter with a few examples, and a particular case.

Example 6.5. Let k be a field. The cross product

$$\bigoplus_{p+q=n} H_p(X;k) \otimes_k H_q(Y;k) \to H_n(X \times Y;k)$$

is an isomorphism. This comes from the fact that Tor^k vanishes over a field (all k-modules are free) and that the first map in the Künneth exact sequence from Theorem 5.4 is precisely the cross product.

Example 6.6. We compute the homology of a torus, and more generally that of a product of spheres. Let $\iota_n \in H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$ and $1_n \in H_0(S^n;\mathbb{Z}) \cong \mathbb{Z}$ denote two generators. Then $H_*(S^m \times S^n;\mathbb{Z})$ consists in four copies of the integers in degree $0, m, n$, and $m+n$ generated by the cross products $1_m \times 1_n, \iota_m \times 1_n, 1_m \times \iota_n$, and $\iota_m \times \iota_n$. Indeed there are no torsion terms in the Künneth formula because the homology of a sphere is torsion free.

CHAPTER 4

Singular cohomology

We start with a few recollections about the definition and elementary properties of singular/cellular cohomology. Our aim in this chapter is then to introduce and study the *cup product* which endows the cohomology $H^*(X;R)$ with coefficients in a ring R with a graded ring structure. For this we will need a cohomological version of the cross product.

1. The cohomology of a space

Let X be a space and R any ring. The standard dualization procedure allows us to define the singular cohomology $H^*(X;R)$ as the cohomology of $\mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X), R)$. There is an alternative way.

Proposition 1.1. *There is a natural isomorphism of R-modules*

$$\eta\colon H^n(X;R) \to H^n(\mathrm{Hom}_R(S_\bullet(X;R), R)$$

PROOF. We have a natural adjunction isomorphism

$$\mathrm{Hom}_R(S_\bullet(X;R), R) \to \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X;\mathbb{Z}), R)$$

where we use the definition of $S_\bullet(X;R) = S_\bullet(X;\mathbb{Z}) \otimes R$, and the identification $R \cong \mathrm{Hom}_R(R,R)$. This isomorphism is compatible with coboundaries, which are defined on the integral singular chain complex and extended R-linearly on $S_\bullet(X;R)$. □

There is an analogous statement for pairs, and we will use, without recalling them here, all standard properties enjoyed by singular cohomology: long exact sequences for pairs, Mayer-Vietoris exact sequences for excisive pairs, and homotopy invariance.

Let us however take some time to deal with the cohomological version of the universal coefficient theorem. We start with the algebraic version. We recall that for a PID all higher Ext^n, for $n > 1$, vanish, just like higher Tor_n do.

Theorem 1.2. *Let R be a PID, S any ring, and $\phi\colon R \to S$ a ring homomorphism. For any chain complex $C_\bullet$ of free R-modules, there is a natural short exact sequence*

$$0 \to \mathrm{Ext}_R(H_{n-1}(C_\bullet), S) \to H^n(\mathrm{Hom}_R(C_\bullet, S)) \xrightarrow{\kappa} \mathrm{Hom}_R(H_n(C_\bullet), S) \to 0$$

which splits, but non-naturally.

Proof. Just like in the homological case we use the short exact sequence

$$0 \to Z_\bullet \to C_\bullet \to B_\bullet[1] \to 0$$

of chain complexes of free R-modules (we assume that R is a PID). We dualize and obtain a short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}_R(B_\bullet[1], S) \to \mathrm{Hom}_R(C_\bullet, S) \to \mathrm{Hom}_R(Z_\bullet, S) \to 0$$

because $\mathrm{Ext}_R(-, S)$ vanishes on a free R-module. For simplicity we call these duals $B^\bullet[-1], C^\bullet$, and $Z^\bullet$, and get a long exact sequence in cohomology. The connecting homomorphisms can be described easily since the differentials on $B_\bullet$ and $Z_\bullet$ are zero. Hence

$$\partial^q\colon H^q(Z^\bullet) = Z^q \to H^{q+1}(B^\bullet[-1]) \cong H^q(B^\bullet) = B^q$$

is induced by the inclusion of boundaries into cycles. The n-th cohomology S-module of $C^\bullet$ is therefore an extension of $\mathrm{Ker}\,\partial^n$ by $\mathrm{Coker}\,\partial^{n-1}$. The "Hom-Ext" exact sequence reduces here to a four term exact sequence since Z_n is free:

$$0 \to \mathrm{Hom}_R(H_n(C_\bullet), S) \to Z^n \to B^n \to \mathrm{Ext}_R(H_n(C_\bullet), S) \to 0$$

which yields the desired short exact sequence. Naturality follows from the naturality of all the steps described above.

We conclude the proof by showing that the short exact sequence admits a splitting. The short exact sequence $Z_n \xrightarrow{i_n} C_n \to B_{n-1}$ admits a splitting since B_{n-1} is free, thus also a retraction $r_n\colon C_n \to Z_n$ such that $r_n \circ i_n = id$. Let $\varphi\colon H_n(C_\bullet) \to S$ be any homomorphism of R-modules. The composition

$$C_n \xrightarrow{r_n} Z_n \twoheadrightarrow Z_n/B_n = H_n(C_\bullet) \xrightarrow{\varphi} S$$

is then an n-cocycle in $C^\bullet$ because $r_n \circ d_{n+1}$ is the corestriction of d_{n+1} onto Z_n as r_n is a retraction of $Z_n \subset C_n$ and the image of the differential is $B_n \subset Z_n$. This construction defines thus a map $s\colon \operatorname{Hom}_R(H_n(C_\bullet), S) \to H^n(C^\bullet)$. It is obviously R-linear and provides a section of the short exact sequence since $H^n(C^\bullet) \to \operatorname{Hom}_R(H_n(C_\bullet), S)$ is induced by the inclusion $Z_n \subset C_n$ of which r_n is a retraction. □

By applying the algebraic universal coefficients Theorem 1.2 to the singular chain complex, we obtain the following cohomological version for spaces.

Theorem 1.3. *Let R be a PID, S a ring with a ring homomorphism $\varphi\colon R \to S$, and X a space. There is a natural short exact sequence*

$$0 \to \operatorname{Ext}_R(H_{n-1}(X; R), S) \to H^n(X; S) \xrightarrow{\kappa} \operatorname{Hom}_R(H_n(X; R), S) \to 0$$

which splits, but non-naturally. □

There are two important cases for us. The first one is the classical case, when $R = \mathbb{Z}$ and S is any ring, where Hom and Ext are given by homomorphisms and extensions of abelian groups. The second one is the case $R = S$.

We conclude this section by recalling that all the previous statements also holds for other versions of ordinary cohomology. We could use for example the cellular chain complex for a CW-complex or the simplicial one for a simplicial complex.

2. The cohomological cross product

In this section we introduce an external product in cohomology. This will follow the method we used to define the cross product in homology. For this we need a homomorphism allowing us to compare the dual of the tensor product of two chain complexes $\operatorname{Hom}_R(C_\bullet \otimes_R D_\bullet, R)$ with the tensor product of the duals. For any p, q, p', q' with $p + q = n = p' + q'$ define

$$\nu\colon \operatorname{Hom}_R(C_p, R) \otimes_R \operatorname{Hom}_R(D_q, R) \to \operatorname{Hom}_R(C_{p'} \otimes_R D_{q'}, R)$$

as follows. For $\varphi\colon C_p \to R$, $\psi\colon D_q \to R$, and $x \in C_{p'}, y \in D_{q'}$, the homomorphism $\nu(\varphi \otimes \psi)$ sends $x \otimes y$ to $\varphi(x) \cdot \psi(y)$ when $p = p'$ and $q = q'$ (and zero otherwise). We compute the coboundary (in the non-trivial case):

$$d[\nu(\varphi \otimes \psi)] = \nu(d\varphi \otimes \psi) + (-1)^p \nu(\varphi \otimes d\psi)$$

The image of a tensor product of two cocycles is therefore a cocycle so that ν defines a homomorphism on cohomology classes

$$\nu\colon H^p(C^\bullet)\otimes_R H^q(D^\bullet)\to H^n(\mathrm{Hom}_R(C_\bullet\otimes_R D_\bullet, R))$$

Let us apply this to the case of singular chain complexes $S_\bullet(X,R)$ and $S_\bullet(Y,R)$. We choose an Alexander-Whitney map $AW\colon S_\bullet(X\times Y,R)\to S_\bullet(X,R)\otimes_R S_\bullet(Y,R)$.

Definition 2.1. Let $p+q=n$. The *cohomological cross product* is the composition

$$H^p(X;R)\otimes_R H^q(Y;R)\xrightarrow{\nu} H^n(\mathrm{Hom}_R(S_\bullet(X,R)\otimes_R S_\bullet(Y,R),R))\xrightarrow{AW^*} H^n(X\times Y;R)$$

We write $\varphi\times\psi = AW^*[\nu(\varphi\otimes\psi)]$.

The Alexander-Whitney map is not unique, but all choices are homotopic, so that the cross product does not depend on the chosen AW map. The properties of the cohomological cross product are analogous to the properties of the homological one. In particular the cross product is natural in both variables X and Y, and the projections induce maps in cohomology whose image is a trivial cross product unless its a cross product with a multiple of the unit in degree zero. More precisely, the second projection $X\times Y\to Y$ induces a map $H^n(Y;R)\to H^n(X\times Y;R)$ that sends a class y to $1\times y$, where $1\in H^0(X;R)$ is the class represented by the augmentation map $S_0(X,R)\to R$ (which is the image of the augmentation map $id\colon S_0(\star,R)=R\to R$). The proofs of these facts and the properties of the next proposition are very similar to the ones we have done in homology.

Proposition 2.2. *The cohomological cross product is associative and commutative in the graded sense.* □

We finish this section by mentioning that there is a relative version of the cross product, which can only be defined for excisive pairs. There is also a Künneth formula in cohomology, but instead of developing this we will perform some computations later by taking advantage of the close relationship between the homological cross product and its cohomological counterpart. We use the same letter for the (co)homology class and for a representing (co)cycle.

Proposition 2.3. *Let* $\varphi \in H^p(X;R)$, $\psi \in H^q(Y;R)$, *and* $x \in H_p(X;R), y \in H_q(Y;R)$. *Then* $(\varphi \times \psi)(x \times y) = \varphi(x) \cdot \psi(y)$.

Proof. We have to evaluate the cocycle $\varphi \times \psi \colon S_n(X \times Y;R) \to R$ on the cycle $x \times y$. We go back to the definitions of both cross products:

$$(\varphi \times \psi)(x \times y) = AW^*[\nu(\varphi \otimes \psi)](EZ_*[\lambda(x \otimes y)])$$

Let us fix for the moment the element $\nu(\varphi \otimes \psi)$ which is represented by a homomorphism in $\mathrm{Hom}_R(\oplus_{p'+q'=n} S_{p'}(X,R) \otimes_R S_{q'}(Y;R), R)$. We precompose this homomorphism with an Alexander-Whitney map to get a homomorphism from $S_n(X \times Y;R)$ which represents $\varphi \times \psi$. To evaluate this cohomology class on $x \times y$ means that we push $\lambda(x \otimes y)$ via an Eilenberg-Zilber morphism to $S_n(X \times Y;R)$, so that we are looking at the (co)chain level at:

$$[\nu(\varphi \otimes \psi)](AW \circ EZ)[\lambda(x \otimes y)]) \simeq [\nu(\varphi \otimes \psi)][\lambda(x \otimes y)]) = \varphi(x) \cdot \psi(y)$$

where the $\simeq$ sign means homotopic (the composite $AW \circ EZ$ induces the identity in homology), and the last equality follows from the definition of ν. In homology we therefore obtain the desired equality. □

Example 2.4. Let u_n denote the degree n generator of $H^n(S^n;\mathbb{Z})$. We already know from the compatibility with the unit, that 1×1, $1 \times u_m$, and $u_n \times 1$ are generators of copies of the integers in degree $0, m$, and n.

We claim now that $u_n \times u_m$ is a generator of $H^{n+m}(S^n \times S^m;\mathbb{Z}) \cong \mathbb{Z}$. This can be seen by a direct computation, or by using the universal coefficient Theorem 1.3, which provides an isomorphism $\kappa \colon H^{n+m}(S^n \times S^m;\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_{n+m}(S^n \times S^m;\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}$. Finally, Proposition 2.3 tells us that the evaluation of $u_n \times u_m$ on the homological cross product $\iota_n \times \iota_m$ is equal to the product $u_n(\iota_n) \cdot u_m(\iota_m) = 1$. Hence $u_n \times u_m$ corresponds to the identity $\mathrm{id}_{\mathbb{Z}}$ in the Hom-group, equivalently a generator in $\mathbb{Z}$.

3. The cup product

The diagonal map $X \to X \times X$ allows us to obtain a cohomology class in $H^*(X;R)$ that comes from the cross product of two cohomology classes. In this section R is a PID.

Definition 3.1. The *cup product* $a \cup b$ of cohomology classes $a \in H^p(X;R)$ and $b \in H^q(X;R)$ is the image of $x \otimes y$ under the composition

$$H^p(X;R) \otimes_R H^q(X;R) \xrightarrow{\times} H^n(X \times X;R) \xrightarrow{\Delta^*} H^n(X;R)$$

where $n = p + q$.

Remark 3.2. Taken from https://hsm.stackexchange.com/questions/3234/who-discovered-the-singular-cup-product. The history of the cup product is described on pages 135–136 of "Never a Dull Moment: Hassler Whitney, Mathematics Pioneer" by Keith Kendig.

At the 1934 International Conference in Topology, held in Moscow, Andrey Kolmogoroff and James Alexander independently presented a talk with an idea how to define a product in cohomology, but it apparently wasn't very precise.

Whitney heard Kolmogoroff and Alexander's definitions of product, and on the one hand the idea seemed tremendously significant, but at the same time something didn't smell quite right. Whitney had made friends with the Czech mathematician Edouard Čech, and the two of them agreed that the multiplication idea seemed powerful and promising.

Nevertheless, there seemed to be a problem: If you multiply two sets like a disk and an interval, the result is a solid cylinder—the dimensions add: 2+1=3. Kolmogoroff and Alexander's definition of product didn't do that, instead giving a dimension that was larger than the sum by one. Whitney and Čech puzzled over this problem, and within a few months they had ironed out the difficulty.

By that time, Alexander had already written a paper using his original definition and submitted it to the Annals. Upon learning of the Whitney–Čech revision, Alexander immediately saw its advantages. He rewrote and resubmitted his paper, which appeared in 1936, [**2**].

An easy consequence of the naturality of the cross product is the naturality of the cup product.

Proposition 3.3. *The cup product is natural: For any map $f\colon X \to Y$, and cohomology classes $a \in H^p(Y;R)$ and $b \in H^q(Y;R)$, we have $f^*(a) \cup f^*(b) = f^*(a \cup b)$.*

PROOF. This follows from the naturality of the cross product as mentioned above, and the naturality of the diagonal map, i.e. $\Delta_Y \circ f = (f \times f) \circ \Delta_Y$. $\square$

The other properties we have seen for the cross product tell us that the cohomology of a space is not only a graded R-module, but comes equipped with a multiplication making it a graded commutative R-algebra.

Proposition 3.4. *The cohomology $H^*(X;R)$ of any space X is a graded commutative R-algebra. The cup product is associative, distributive with respect to the sum, compatible with multiplication by scalars, and commutative in the graded sense.*

PROOF. Associativity follows from the associativity of the cross product and the fact that the triple diagonal Δ_3 can be written $\Delta \circ (\Delta \times X)$ or $\Delta \circ (X \times \Delta)$. The following diagram is commutative by naturality of the cross product (we drop the coefficients in the notation to make it more compact):

$$\begin{array}{ccc} H^{p+q}(X \times X) \otimes H^r X & \xrightarrow{\Delta^* \otimes Id} & H^{p+q} X \otimes H^r X \\ \downarrow \times & & \downarrow \times \\ H^{p+q+r}(X \times X \times X) & \xrightarrow{(\Delta \times Id)^*} & H^{p+q+r}(X \times X) \end{array}$$

This implies that the image of $(a \times b) \otimes c$ under the "top-right" composition, namely $(a \cup b) \times c$ coincides with the image going the other way, i.e. $(\Delta \times X)^*[(a \times b) \times c]$. Pushing this class in $H^{p+q+r}X$ under Δ^* shows that

$$(a \cup b) \cup c = (\Delta_3)^*[(a \times b) \times c]$$

By associativity of the cross product this is the same as $(\Delta_3)^*[a \times (b \times c)]$, and the same argument identifies then in turn this element with $a \cup (b \cup c)$.

The compatibility with scalar multiplication and distributivity are direct consequences of the analogous properties for the cross product. So is graded commutativity since $T \circ \Delta = \Delta$ and thus

$$b \cup a = \Delta^*(b \times a) = \Delta^*[T^*(b \times a)] = \Delta^*[(-1)^{pq}(a \times b)] = (-1)^{pq}(a \cup b)$$

Finally the unit for this product is $1 \in H^0(X;R)$ represented by the augmentation map $S_0(X,R) \to R$. Indeed, as $p_2 \circ \Delta = Id_X$,

$$1 \cup b = \Delta^*(1 \times b) = \Delta^*[(p_2)^*(b)] = b$$

The same computation shows that $a \cup 1 = a$, and this concludes the proof. $\square$

Before doing our first computations of cup products, let us note the following compatibility between cup and cross products.

Proposition 3.5. *Let $x \in H^p(X;R)$, $x' \in H^q(X;R)$, $y \in H^r(Y;R)$, $y' \in H^s(Y;R)$. Then $(x \times y) \cup (x' \times y') = (-1)^{qr}(x \cup x') \times (y \cup y')$.*

Proof. The cup product of cross product is the image under the diagonal map of the cross product $(x \times y) \times (x' \times y') \in H^{p+q+r+s}(X \times Y \times X \times Y; R)$. We can also write the diagonal map of $X \times Y$ as the composition

$$X \times Y \times X \times Y \xrightarrow{X \times T \times Y} X \times X \times Y \times Y \xrightarrow{\Delta_X \times \Delta_Y} X \times Y$$

Under this composition, using the graded commutativity of the cross product, we see that the class $(x \times y) \cup (x' \times y')$ is equal to $(-1)^{qr}(x \cup x') \times (y \cup y')$. $\square$

Example 3.6. Let $X = S^n \times S^m$. We wish to determine the graded $\mathbb{Z}$-algebra structure of $H^*(X;\mathbb{Z})$. Let x and y denote the generators in degree n and m, images of the corresponding classes $u_n \in H^n(S^n;\mathbb{Z})$ and $u_m \in H^m(S^m;\mathbb{Z})$ under the projection maps $p_1 \colon X \to S^n$ and $p_2 \colon X \to S^m$. We claim that $x \cup y$ is a generator of $H^{n+m}(S^n \times S^m;\mathbb{Z})$.

The class x is actually a cross product $u_n \times 1$ and likewise $y = 1 \times u_m$. Therefore, by Proposition 3.5, we see that

$$x \cup y = (u_n \times 1) \cup (1 \times u_m) = (u_n \cup 1) \times (1 \cup u_m) = u_n \times u_m$$

This cross product is a generator of $H^{n+m}(S^n \times S^m;\mathbb{Z})$ as shown in Example 2.4 This can be seen by a direct computation, or by using the universal coefficient Theorem 1.3. It provides an isomorphism

$$H^{n+m}(S^n \times S^m;\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_{n+m}(S^n \times S^m;\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}$$

since Proposition 2.3 tells us that the evaluation of $u_n \times u_m$ on the homological cross product $\iota_n \times \iota_m$ (our favorite generator in degree $n+m$) is equal to the product $u_n(\iota_n) \cdot u_m(\iota_m) = 1$.

Hence $H^*(S^n \times S^m;\mathbb{Z})$ is isomorphic to an exterior algebra $\Lambda(x_n, y_m)$ on two generators of degree n and m respectively.

4. The cohomology of projective spaces

At this moment of the course we have finally introduced the cup product in singular cohomology, this was our objective from the start. But just like the cup product in group cohomology we do not claim that the identification of this multiplicative structure is easy in general. In this section we illustrate this with a fundamental example, that of real and complex projective spaces, where the algebra structure is highly non-trivial. We follow Hatcher's approach in [**14**] because it is really an illustration of how the definition of the cup product works. There are more sophisticated ways to obtain the same result, see for example [**15**]: they first introduce Poincaré duality for manifolds and rely on the fact that $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are manifolds. We will instead only use an explicit description of attaching maps and compare the cup product we are interested in with the relative one established in an exercise last week. We had done it for pairs consisting of cubes and their boundaries, now we will use the following variant.

Lemma 4.1. *Let $n = i + j$ with $i, j \geq 1$. The relative cup product*

$$H^j(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^i) \otimes H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^j) \to H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$$

is an isomorphism with integral or mod p coefficients.

Proof. The pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^i)$ is homotopy equivalent to $(\mathbb{R}^j, \mathbb{R}^j \setminus 0)$ by projecting onto the last j coordinates if we view the inclusion $\mathbb{R}^i \subset \mathbb{R}^n$ as given by the formula

$$(x_1, \ldots, x_i) \mapsto (x_1, \ldots, x_i, 0, \ldots, 0)$$

The latter pair is in turn homologically equivalent to the subpair $(I^j, \dot{I}^j)$ (but not homotopy equivalent as pairs, something I might have said too quickly in the 2023 lecture you might have watched on video). The same holds for the projection onto the last j components and we notice that the chosen copies of $\mathbb{R}^i$ and $\mathbb{R}^j$ intersect only in the origin 0 so that the relative cup product lands into $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ as claimed. All together this explains why the relative cup product we are considering here is equivalent to the one on cubes from Exercise 2, Sheet 11.

In the absence of torsion the relative *cross* product

$$H^j(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^i) \otimes H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^j) \to H^n(\mathbb{R}^n \times \mathbb{R}^n, (\mathbb{R}^n \setminus \mathbb{R}^i) \times \mathbb{R}^n \cup \mathbb{R}^n \times (\mathbb{R}^n \setminus \mathbb{R}^j))$$

is an isomorphism (Exercise 1, Sheet 10). The conclusion then follows since the composition below is the identity:

$$(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \xrightarrow{\Delta} (\mathbb{R}^n \times \mathbb{R}^n, (\mathbb{R}^n \setminus \mathbb{R}^i) \times \mathbb{R}^n \cup \mathbb{R}^n \times (\mathbb{R}^n \setminus \mathbb{R}^j)) \xrightarrow{p} (\mathbb{R}^n, \mathbb{R}^n \setminus 0)$$

where Δ is the diagonal map, and p projects onto the coordinates $i+1, \ldots, n$ and $n+1, \ldots, n+i$. The second map is a strong deformation retraction, thus the diagonal map is a homotopy equivalence (it is in fact good enough to observe that these maps induce an equivalence on relative homology groups). □

Before moving to the general computation for $\mathbb{R}P^n$, let us illustrate these methods in the case $n = 2$ where we can draw the situation. We view $\mathbb{R}P^2$ as the quotient of $S^2 \subset \mathbb{R}^3$ by the antipodal relation and write (x_0, x_1, x_2) the coordinates of a point in $\mathbb{R}^3$.

Proposition 4.2. *The mod 2 cohomology of $\mathbb{R}P^2$ is truncated polynomial $\mathbb{F}_2[u]/(u^3)$.*

Proof. We know that $H^0(\mathbb{R}P^2; \mathbb{F}_2) \cong \mathbb{F}_2$ is generated by the unit 1 in cohomology. Let us call $u \in H^1(\mathbb{R}P^2; \mathbb{F}_2) \cong \mathbb{F}_2$ the generator. We also know that $H^2(\mathbb{R}P^2; \mathbb{F}_2) \cong \mathbb{F}_2$ and all higher cohomology groups are zero for dimensional reasons (the cellular cochain complex is zero in degrees ≥ 3). The only computation we have to make is therefore $u \cup u$.

Let us call P and P' the subspaces of $\mathbb{R}P^2$ consisting in the quotients of the circles in S^2 having $x_2 = 0$ and respectively $x_0 = 0$. The intersection $P \cap P'$ is a single point e_1 represented by $(0, 1, 0)$. The pair $(\mathbb{R}P^2, \mathbb{R}P^2 \setminus P)$ induces a long exact sequence in cohomology. Since $\mathbb{R}P^2 \setminus P$ is homeomorphic to an open cell (the one one attaches to the projective circle P to build the projective plane), we have isomorphisms $H^1(\mathbb{R}P^2, \mathbb{R}P^2 \setminus P; \mathbb{F}_2) \cong H^1(\mathbb{R}P^2; \mathbb{F}_2)$ and the same holds for the other projective circle P'. Likewise, when considering the pair $(\mathbb{R}P^2, \mathbb{R}P^2 \setminus e_1)$ we see that $H^2(\mathbb{R}P^2, \mathbb{R}P^2 \setminus e_1; \mathbb{F}_2) \cong H^2(\mathbb{R}P^2, \mathbb{R}P^1; \mathbb{F}_2) \cong H^2(\mathbb{R}P^2; \mathbb{F}_2) \cong \mathbb{F}_2$ (or use cellular cohomology).

Therefore, by naturality, instead of computing the absolute cup product

$$H^1(\mathbb{R}P^2; \mathbb{F}_2) \otimes H^1(\mathbb{R}P^2; \mathbb{F}_2) \to H^2(\mathbb{R}P^2; \mathbb{F}_2)$$

we can do the equivalent computation of a relative cup product

$$H^1(\mathbb{R}P^2, \mathbb{R}P^2 \setminus P; \mathbb{F}_2) \otimes H^1(\mathbb{R}P^2, \mathbb{R}P^2 \setminus P'; \mathbb{F}_2) \to H^2(\mathbb{R}P^2, \mathbb{R}P^2 \setminus e_1; \mathbb{F}_2)$$

As a last step we apply excision to the CW-pair $(\mathbb{R}P^2, \mathbb{R}P^2 \setminus e_1)$ by excising a third copy of $\mathbb{R}P^1$, namely the one with $x_1 = 0$. We obtain the pair consisting in the top cell of the projective plane $(\mathbb{R}P^2 \setminus \mathbb{R}P^1, \mathbb{R}P^2 \setminus (\mathbb{R}P^1 \cup e_1)) \approx (\mathbb{R}^2, \mathbb{R}^2 \setminus 0)$. This yields precisely the non-trivial relative cup product from Lemma 4.1:

$$H^1(\mathbb{R}^2, \mathbb{R}^2 \setminus \mathbb{R}) \otimes H^1(\mathbb{R}^2, \mathbb{R}^2 \setminus \mathbb{R}') \to H^2(\mathbb{R}^2, \mathbb{R}^2 \setminus 0)$$

where we used the notation with a prime to distinguish the two different copies of $\mathbb{R} \subset \mathbb{R}^2$. This shows that the top class in degree 2 is the cup product $u \cup u$, which concludes the proof. □

We follow the same strategy to obtain the ring structure of the mod 2 cohomology of all real projective spaces.

Theorem 4.3. *The mod 2 cohomology of $\mathbb{R}P^n$ is truncated polynomial $\mathbb{F}_2[u]/(u^{n+1})$ on a generator u of degree one, and $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[u]$.*

Proof. We proceed by induction, the initialization being done with $\mathbb{R}P^1$, and even the case of $\mathbb{R}P^2$ in Proposition 4.2. Assume therefore that we have an isomorphism of $\mathbb{F}_2$-algebras $H^*(\mathbb{R}P^{n-1}; \mathbb{F}_2) \cong \mathbb{F}_2[u]/(u^n)$. The inclusion $\mathbb{R}P^{n-1} \subset \mathbb{R}P^n$ induces a homomorphism $H^*(\mathbb{R}P^n; \mathbb{F}_2) \to H^*(\mathbb{R}P^{n-1}; \mathbb{F}_2)$, which is an isomorphism in degrees $< n$. Therefore, if u denotes the non-zero class in degree 1 we already know that u^k is non-zero for all $1 \leq k \leq n-1$ because its image is non-zero.

It is therefore good enough to show that the class $u^{n-1} \cup u$ is non-trivial in $H^n(\mathbb{R}P^n; \mathbb{F}_2)$. The n-dimensional real projective space is a quotient of $S^n \subset \mathbb{R}^{n+1}$, a point is therefore represented by a norm one element $(x_0, \dots, x_n)$. We let P^1 denote the subspace of $\mathbb{R}P^n$ consisting of elements of the form $(x_0, x_1, 0, \dots, 0)$ (where we do not distinguish notationally elements in the sphere and the classes they represent in the projective space). This space P^1 is homeomorphic to $\mathbb{R}P^1$, and P^{n-1}, which is homeomorphic to $\mathbb{R}P^{n-1}$, correspond to elements of the form $(0, x_1, \dots, x_n)$. The intersection of these subspaces consists in e_1, the class of $(0, 1, 0, \dots, 0)$. The corresponding linear subpaces of $\mathbb{R}^n$ are called $\mathbb{R}^1$ and $\mathbb{R}^{n-1}$ respectively. We consider the following commutative diagram of horizontal cup products (by naturality of the relative cup product):

$$\begin{array}{ccc} H^{n-1}(\mathbb{R}P^n;\mathbb{F}_2)\otimes H^1(\mathbb{R}P^n;\mathbb{F}_2) & \longrightarrow & H^n(\mathbb{R}P^n;\mathbb{F}_2) \\ \uparrow & & \uparrow \\ H^{n-1}(\mathbb{R}P^n,\mathbb{R}P^n\setminus P^1;\mathbb{F}_2)\otimes H^1(\mathbb{R}P^n,\mathbb{R}P^n\setminus P^{n-1};\mathbb{F}_2) & \longrightarrow & H^n(\mathbb{R}P^n,\mathbb{R}P^n\setminus e_1;\mathbb{F}_2) \\ \downarrow & & \downarrow \\ H^{n-1}(\mathbb{R}^n,\mathbb{R}^n\setminus\mathbb{R}^1)\otimes H^1(\mathbb{R}^n,\mathbb{R}^n\setminus\mathbb{R}^{n-1}) & \longrightarrow & H^n(\mathbb{R}^n,\mathbb{R}^n\setminus 0) \end{array}$$

We have isomorphisms on the right hand side just as in the the previous proposition, but the situation on the left is slightly more complicated now. Let us deal with the maps with $H^{n-1}(\mathbb{R}P^n,\mathbb{R}P^n\setminus P^1;\mathbb{F}_2)$ as domain. The points in the difference $\mathbb{R}P^n\setminus P^1$ are of the form $(x_0,\dots,x_n)$ with at least one of the coordinates $x_2,\dots,x_n$ non zero. There is a homotopy which at time t sends such a point to $(tx_0,tx_1,x_2,\dots,x_n)$ divided by its norm (which is always non-zero). This shows that $P^{n-2}\approx\mathbb{R}P^{n-2}$ is a deformation retract of $\mathbb{R}P^n\setminus P^1$. The long exact sequence in cohomology of the pair $(\mathbb{R}P^n,P^{n-2})$ shows then that the top left vertical map is an isomorphism.

Alternatively, more in the spirit of the computation done for $\mathbb{R}P^2$ and easier to use to show not only that we have vertical isomorphisms, but also a commutative square, we can use excision. To be left with an open disk we excise a copy of $\mathbb{R}P^{n-1}$, namely the one coming from points of the form $(x_0,0,x_2,\dots,x_n)\in S^n$. We do so in each of the cohomology groups of pairs in the middle row simultaneously and precisely chose this copy so that the point e_1 does not belong to it.

We finally identify the bottom left vertical map as an isomorphism by projecting $\mathbb{R}^n$ onto its last coordinates $\mathbb{R}^{n-1}$, so that

$$H^{n-1}(\mathbb{R}^n,\mathbb{R}^n\setminus\mathbb{R}^1;\mathbb{F}_2)\cong H^{n-1}(\mathbb{R}^{n-1},\mathbb{R}^{n-1}\setminus 0;\mathbb{F}_2)$$

We have already seen that the latter is isomorphic to $H^{n-1}(\mathbb{R}P^{n-1},\mathbb{R}P^{n-1}\setminus e_1;\mathbb{F}_2)$ and this in turn is the same as $H^{n-1}(\mathbb{R}P^{n-1},\mathbb{R}P^{n-2};\mathbb{F}_2)$, which is isomorphic to $H^{n-1}(\mathbb{R}P^n,\mathbb{R}P^n\setminus P^1;\mathbb{F}_2)$ by the previous deformation retraction argument.

The bottom cup product is an isomorphism by Lemma 4.1, hence so is the top cup product, which shows that $u^{n-1}\cup u$ is a generator of $H^n(\mathbb{R}P^n;\mathbb{F}_2)$. $\square$

We state now the analogous results for complex and quaternionic projective spaces.

Theorem 4.4. *The integral cohomology of* $\mathbb{C}P^n$ *is truncated polynomial* $\mathbb{Z}[u]/(u^{n+1})$ *on a generator* u *of degree two, and* $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[u]$*. The integral cohomology of* $\mathbb{H}P^n$ *is truncated polynomial* $\mathbb{Z}[u]/(u^{n+1})$ *on a generator* u *of degree four, and* $H^*(\mathbb{H}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[u]$*.*

It is interesting to notice that the lack of associativity of the octonionic multiplication allows one to define an octonionic projective plane $\mathbb{O}P^2$, but higher dimensional projective spaces do not exist.

Let us conclude this section by an illustration of what we can do with the extra multiplicative structure on the cohomology of a space.

Example 4.5. Let $X = S^2 \vee S^4 \vee S^6$, $Y = \mathbb{C}P^3$, and $Z = \mathbb{C}P^2 \vee S^6$. All three spaces are simply connected, they have the same homology and cohomology with integral coefficients, as graded vector spaces, but they have different homotopy types since the $\mathbb{Z}$-algebra structures on $H^*(-;\mathbb{Z})$ are different.

In $H^*(X;\mathbb{Z})$ all cup products vanish because every sphere in the wedge decomposition is a retract. The cohomology $H^*(Y;\mathbb{Z})$ is truncated polynomial, and in $H^*(Z;\mathbb{Z})$ a degree four generator is a cup product, but a degree six generator is not.

5. Comparison with the cup product in group cohomology

In this final section we wrap up the discussion by explaining that the two cup products we have seen in this course are actually the same. This does not come as a surprise since we already know that group cohomology can be seen as singular cohomology of the classifying space: $H^*(G;\mathbb{Z}) \cong H^*(BG;\mathbb{Z})$. For simplicity we only deal with integral (and trivial) coefficients).

Proposition 5.1. *The group theoretical cup product on* $H^*(G;\mathbb{Z})$ *agrees with the cup product on singular cohomology* $H^*(BG;\mathbb{Z})$*.*

Proof. To compare group (co)homology with the singular version, we have used a specific classifying space model, based on the simplicial object $G^{\bullet+1}$ on which G acts diagonally. Its geometric realization is a contractible CW-complex EG equipped with a free action of G on the cells and $BG = EG/G$, see Definition 7.3. The comparison of the seemingly different homology theories was then obtained in Corollary 7.5 by noticing that the cellular chain complex of EG forms a free $\mathbb{Z}G$-resolution $F_\bullet$ of $\mathbb{Z}$, namely the bar resolution, and that of BG is precisely $F_\bullet \otimes_{\mathbb{Z}G} \mathbb{Z}$.

This dualizes easily so that $H^n(G;\mathbb{Z})$ is computed by the cellular cochain complex of BG, which agrees with singular cohomology as we know. Moreover, both cup products are defined as Δ^* of a cross product, where Δ is the diagonal. Observe that the diagonal $G \to G \times G$ is a group homomorphism, it thus induces a map $EG \to E(G \times G)$ which is compatible with the respective actions of G and $G \times G$. It then yields a map $BG \to B(G \times G)$. Both projections to BG define a map $B(G \times G) \to BG \times BG$ which is a homotopy equivalence (both spaces are models of $B(G \times G)$, one is a quotient of $E(G \times G)$, the other one of $EG \times EG$).

Therefore, the composition $BG \xrightarrow{B\Delta} B(G \times G) \to BG \times BG$ being the topological diagonal map, the homomorphisms Δ^* on $H^*(G;\mathbb{Z})$ and Δ^* on $H^*(BG;\mathbb{Z})$ coincide. It is thus sufficient to compare the group theoretical and the topological cross products.

Recall that in group cohomology we start with a free $\mathbb{Z}G$-resolution $F_\bullet$ of $\mathbb{Z}$ (for example the bar resolution), and dualize to get $\mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, \mathbb{Z})$. The cross product of two classes represented by cocycles $u \in \mathrm{Hom}_{\mathbb{Z}G}(F_p, \mathbb{Z})$ and $v \in \mathrm{Hom}_{\mathbb{Z}G}(F_q, \mathbb{Z})$ is represented by the n-cocycle given by $u \otimes v\colon F_p \otimes F_q \to \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$ (extended by zero on factors $F_i \otimes F_j$ for different pairs (i,j) than (p,q) with $i+j = n = p+q$. When interpreting this in the case of the bar resolution $F_\bullet$ and viewing it as the cellular chain complex of EG, we see that there is no need to use an Alexander-Whitney map because $F_\bullet \otimes F_\bullet$ is *isomorphic* to the cellular chain complex of $EG \times EG$.

To compare this with the singular version, we take BG to be the classifying space as constructed above, quotient of a free and cellular action on EG. Then the singular chain complex $S_\bullet(EG)$ is augmented over $\mathbb{Z}$ and yields a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, much larger than $F_\bullet$, which is the cellular version. But there is always a chain map $f\colon S_\bullet(EG) \to F_\bullet$, unique up to chain homotopy. The composition

$$S_\bullet(E(G \times G)) \to S_\bullet(EG \times EG) \xrightarrow{AW} S_\bullet(EG) \otimes S_\bullet(EG) \xrightarrow{f \otimes f} F_\bullet \otimes F_\bullet$$

is a chain equivalence and respects the action of $G \times G$ by naturality of the Alexander-Whitney map. Next, since $u \circ f$ and $v \circ f$ are $\mathbb{Z}G$-module homomorphisms, they factor through $S_\bullet(EG)/G \cong S_\bullet(BG)$. The induced map $\overline{u}\colon S_p(BG) \to \mathbb{Z}$ represent then the same cohomology class as u and the same is true for $\overline{v}$.

Finally, by naturality of AW again, we see that the above composition induces the cross product $\overline{u} \times \overline{v}\colon S_n(BG \times BG) \to \mathbb{Z}$. By uniqueness up to homotopy we conclude that this cross product in singular cohomology corresponds precisely to the one in group cohomology. □

Remark 5.2. We understand now that we have done the same computation twice. The mod 2 cohomology of the cyclic group C_2 is polynomial on one generator of degree 1, and so is $H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$. This is simply a consequence of the fact that the infinite real projective space is a BC_2: It can be seen as the quotient of the antipodal action of C_2 on S^∞, a nice and small model of EC_2.

CHAPTER 5

The Steenrod algebra

The cup product has been seen to be an efficient tool to distinguish different homotopy types when the homology or cohomology alone, as a graded module, was not able to see the difference. However the cup product is not sufficient either. In this chapter we enrich the algebra structure with a new layer of algebraic structure, namely that of cohomology operations. This will allow us to distinguish for example the suspension $\Sigma\mathbb{C}P^2$, which shares the same cohomology ring with $S^3 \vee S^5$.

For simplicity we will only deal with the field $\mathbb{F}_2$ of two elements, but there exists an analogous description over the field $\mathbb{F}_p$ for any prime p.

1. Cohomology operations

In this section we introduce the concept of cohomology operation and provide a few examples.

Definition 1.1. Let m, n be two natural integers. A natural transformation

$$\theta\colon H^n(-;\mathbb{F}_2) \to H^m(-;\mathbb{F}_2)$$

is called a *cohomology operation of type* (n, m).

One trivial example is the identity, of type (n, n) for any integer $n \geq 0$. One less trivial example is the cup product $H^n(X;\mathbb{F}_2) \to H^{2n}(X;\mathbb{F}_2)$ sending a class to its square $x^2 = x \cup x$. Here is a new example, the Bockstein homomorphism.

Example 1.2. The short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ induces a short exact sequence of singular cochain complexes

$$0 \to S^\bullet(X;\mathbb{Z}/2) \to S^\bullet(X;\mathbb{Z}/4) \to S^\bullet(X;\mathbb{Z}/2) \to 0$$

There is no Ext-term since the singular chain complex is free over $\mathbb{Z}$. This yields a long exact sequence in cohomology and we are interested in the connecting homomorphisms. Before writing down the definition we note that the first connecting homomorphism $H^0(X;\mathbb{Z}/2) \to H^1(X;\mathbb{Z}/2)$ is trivial: we get the zero homomorphism since the degree zero part provides a short exact sequence.

Definition 1.3. The *Bockstein homomorphisms* are the connecting homomorphisms $\beta\colon H^n(X;\mathbb{F}_2) \to H^{n+1}(X;\mathbb{F}_2)$, for all $n \geq 1$.

Let us try to understand what the Bockstein can detect.

Example 1.4. Let $M(\mathbb{Z}/2^k, n) = S^n \cup e^{n+1}$ where the attaching map of the top cell has degree 2^k for some integer $k \geq 1$. When $k = 1$ the *Moore space* $M(\mathbb{Z}/2, n)$ is a suspension $\Sigma^{n-1}\mathbb{R}P^2$ and we compute $H^n(M(\mathbb{Z}/2,n);\mathbb{F}_2) \cong \mathbb{F}_2 \cong H^{n+1}(M(\mathbb{Z}/2,n);\mathbb{F}_2)$. The long exact sequence in homology we have introduced above starts then like this in degree n:

$$H^{n-1}((M(\mathbb{Z}/2,n);\mathbb{F}_2) = 0 \xrightarrow{\beta} \mathbb{F}_2 \to \mathbb{F}_2 \to \mathbb{F}_2 \xrightarrow{\beta} \mathbb{F}_2 \to \dots$$

The Bockstein homomorphism $\beta\colon H^n(M(\mathbb{Z}/2,n);\mathbb{F}_2) \to H^{n+1}(M(\mathbb{Z}/2,n);\mathbb{F}_2)$ (the only interesting one) sends thus a generator x in degree n to a generator βx in degree $n+1$. It seems that the Bockstein detects the presence of a copy of $\mathbb{Z}/2$ in integral (co)homology and pairs up the resulting two copies of $\mathbb{F}_2$ in mod 2 cohomology.

Let us look now at $M(\mathbb{Z}/2^k, n)$ for $k > 1$. The cellular cochain complex has now trivial differential with $\mathbb{Z}/4$ coefficients, so that the cohomology groups are

$$H^n(M(\mathbb{Z}/2,n);\mathbb{Z}/4) \cong \mathbb{Z}/4 \cong H^{n+1}(M(\mathbb{Z}/2,n);\mathbb{Z}/4)$$

The long exact sequence in cohomology splits into two short exact sequences now, so the Bockstein is zero here. When there is a $\mathbb{Z}/4$ in integral (co)homology, or even higher 2-torsion, two copies of $\mathbb{F}_2$ still appear in mod 2 cohomology by the universal coefficients Theorem, but now the Bockstein does not pair them up.

Let us prove one key feature of the Bockstein before moving to more general cohomology operations.

Proposition 1.5. *The square of the Bockstein homomorphism is always zero:* $\beta^2 x = 0$ *for any cohomology class* $x \in H^n(X; \mathbb{F}_2)$.

Proof. This comes from the fact that there is an "integral" Bockstein homomorphism $B\colon H^n(X;\mathbb{F}_2) \to H^{n+1}(X;\mathbb{Z})$ coming from the short exact sequence $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2$. Let us denote by $\rho\colon H^{n+1}(X;\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}/2)$ the mod 2 reduction. The obvious map between the two short exact sequences of coefficients induces a commutative ladder between the associated long exact sequences, so in particular $\rho \circ B = \beta$. Hence $\beta \circ \beta = \rho \circ B \circ \rho \circ B = 0$ since the middle composition $B \circ \rho$ is zero, as being part of the new long exact sequence. □

This fact tells us that there is a cochain complex

$$\dots \xrightarrow{\beta} H^{n-1}(X;\mathbb{F}_2) \xrightarrow{\beta} H^n(X;\mathbb{F}_2) \xrightarrow{\beta} H^{n+1}(X;\mathbb{F}_2) \xrightarrow{\beta} \dots$$

One can then define the "Bockstein cohomology" as the cohomology groups of this cochain complex. This is in fact only the first step of an iterative process called the Bockstein spectral sequence, which starts from the mod 2 cohomology of a space and pairs up at each stage two copies of $\mathbb{F}_2$ coming from a single copy of $\mathbb{Z}/2^k$ in integral cohomology. At the end only the isolated copies of $\mathbb{F}_2$ coming from non-torsion elements remain...

2. Construction of the Steenrod squares

We have introduced arbitrary cohomology operations in the previous section, but we are interested in stable ones, namely those that are compatible with the suspension isomorphisms in cohomology. Recall that these suspension homomorphisms provide isomorphisms $\sigma\colon H^n(X;\mathbb{F}_2) \to H^{n+1}(\Sigma X;\mathbb{F}_2)$ for all $n \geq 1$.

Definition 2.1. A sequence of cohomology operations θ_n of type $(n, n+k)$ forms a *stable cohomology operation of degree* k if $\theta_{n+1} \circ \sigma = \sigma \circ \theta_n$.

The main idea is due to Steenrod, [**25**], who was able to construct all stable cohomology operations. The reason behind the existence of the Steenrod operations is that the cup product is commutative in cohomology, but not so at the cochain level. We recall that the cup product is induced by the composition

$$S^\bullet(X) \otimes S^\bullet(X) \xrightarrow{AW^*} S^\bullet(X \times X) \xrightarrow{\Delta^*} S^\bullet(X)$$

where all coefficients are $\mathbb{F}_2$ (and left implicit from the notation). The Alexander-Whitney map is a natural transformation, but it is not compatible with the C_2-action given by exchanging the two copies of the (tensor) product. In other words, it does not induce a map on coinvariants $(S^\bullet(X) \otimes S^\bullet(X))_{C_2} \to S^\bullet(X)$.

There is a way to fix this, up to homotopy and all higher homotopies. The tricky part is to construct a sort of C_2-equivariant analogue of the diagonal approximation map. Let $F_\bullet$ be our favorite acyclic augmented free $\mathbb{Z}C_2$-chain complex, namely the periodic one with $F_n = \mathbb{Z}C_2$ and we call g the generator of C_2. We will consider it as a cochain complex by viewing the degree n part as living in degree $-n$. We are looking for what Lurie calls a symmetric multiplication in [**16**], i.e. a cochain map

$$S^\bullet(X) \otimes S^\bullet(X) \otimes F^\bullet \to S^\bullet(X)$$

which is invariant under the C_2-action acting diagonally on the left by permuting the two copies of the singular cochain complex. We will not construct this map here, but let us say that in degree zero we have two multiplications on singular cochains ($a \cdot b$ and $b \cdot a$), induced by the diagonal map and AW. They define a map $S^0(X) \otimes S^0(X) \otimes F^0 \to S^0(X)$ by choosing the first one on the copy tensored with $\mathbb{Z}\,1$ and the second one on $\mathbb{Z}\,g$. They are not equal, but homotopic, and we see on the two cells of F^k, for $k \geq 1$, higher homotopies between homotopies, chosen coherently.

The symmetric multiplication induces then a map on the orbits

$$D_2(S^\bullet(X)) = (S^\bullet(X) \otimes S^\bullet(X) \otimes F^\bullet)_{C_2} \to S^\bullet(X)$$

The cochain complex $D_2(S^\bullet(X))$ is called the (second) *extended power* of $S^\bullet(X)$, or the "quadratic construction" by Jean Lannes in his 2023 EPFL talk where he considered the topological analogue $\mathcal{S}_2(X) = (X^2 \times EC_2)_{C_2}$ and EC_2 can be chosen as the infinite sphere S^∞ with exactly two cells in every dimension that are swapped by the C_2-action.

To every cohomology class u in $H^*(X)$ we now associate a cohomology class Pu in $H^*(D_2(S^\bullet(X)))$. Let us go back to the cochain level. The cohomology class u is represented by a cocycle u in a certain degree n, which can be represented by a cochain map $\mathbb{F}_2[-n] \to S^\bullet(X)$ where the source is the cochain complex concentrated in degree n and consisting there in a single copy of $\mathbb{F}_2$. The commutativity of the square made of the complexes in degree n and $n+1$ says exactly that $du = 0$, this is the cocycle condition.

We then apply the quadratic construction to this map and obtain an extended power map $D_2(u)\colon D_2(\mathbb{F}_2[-n]) \to D_2(S^\bullet(X))$. Let us look at the source:

$$D_2(\mathbb{F}_2[-n]) = (\mathbb{F}_2[-n] \otimes \mathbb{F}_2[-n] \otimes F^\bullet)_{C_2}$$

The first tensor product $\mathbb{F}_2[-n] \otimes \mathbb{F}_2[-n]$ is simply a copy of $\mathbb{F}_2[-2n]$ and the interchange action is the trivial action (no sign issues mod 2). Thus $D_2(\mathbb{F}_2[-n])$ is a shifted copy of $(F^\bullet)_{C_2}$, the mod 2 cellular cochain complex of $\mathbb{R}P^\infty$, aka BC_2. It consists of a copy of $\mathbb{F}_2$ in every degree, all differentials are trivial. The shift of degree $2n$ pushes this cochain complex in degrees $\leq 2n$.

As we have a symmetric multiplication we can further push this construction into the singular cochain complex via

$$D_2(\mathbb{F}_2[-n]) = (\mathbb{F}_2[t])[-2n] \xrightarrow{D_2(u)} D_2(S^\bullet(X)) \xrightarrow{\mu} S^\bullet(X)$$

The element t^{n-i} lives in degree $2n-(n-i) = n+i$ and its image is a cocycle in that same degree, representing a class $Sq^i(u) \in H^{n+i}(X;\mathbb{F}_2)$. This is the way Lurie presents the construction, alternatively we could follow Steenrod's original and equivalent construction. From the cohomology class u in degree n he constructs a $2n$-cocycle $S_\bullet(X) \otimes S_\bullet(X) \otimes F_\bullet \to \mathbb{F}_2$ by projecting via the augmentation of $F_\bullet$ and using the cup square product u^2. This represents the class $Pu \in H^{2n}(\mathcal{S}_2(X);\mathbb{F}_2)$. The diagonal inclusion is C_2-equivariant and induces a map

$$X \times BC_2 = (X \times EC_2)_{C_2} \to \mathcal{S}_2(X)$$

In cohomology the image of Pu is a class in

$$H^{2n}(X \times BC_2;\mathbb{F}_2) \cong \bigoplus H^{n+i}(X;\mathbb{F}_2) \otimes H^{n-i}(BC_2;\mathbb{F}_2)$$

The image is of the form $\sum Sq^i u \otimes t^{n-i}$ and this formula defines the Steenrod squares.

Example 2.2. The zeroth Steenrod square Sq^0 is the identity. The first Steenrod square Sq^1 coincides with the Bockstein.

The expected stability property holds for all Sq^i's.

Proposition 2.3. *The Steenrod squares Sq^i are stable cohomology operations.*

Even more is true. Steenrod constructed in fact all stable cohomology operations. Henri Cartan was the one to call the algebra of all stable cohomology operations the *Steenrod algebra*, in 1955, see [**8**].

Theorem 2.4. *The Steenrod squares generate the graded $\mathbb{F}_2$-algebra of all stable cohomology operations $\mathcal{A}_2$. The unit is $1 = Sq^0$, the degree of Sq^i is i, and the multiplication is given by composition.*

This means that this algebra can be thought of as a quotient of the free algebra generated by the Steenrod squares. However it is not free and José Adem proved that the *Adem relations* describe precisely all existing relations between the Sq^i's:

$$Sq^i Sq^j = \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

when $i < 2j$, [**1**]. For example $Sq^1 Sq^1 = 0$ (the Bockstein is a differential), and $Sq^1 Sq^2 = Sq^3$. In the early sixties Steenrod and Epstein characterized the Steenrod algebra in an axiomatic way, [**25**]. The Steenrod squares are natural $\mathbb{F}_2$-linear transformations, Sq^0 is the identity, but they are not homomorphisms of graded rings, they follow the so-called *Cartan formula*:

$$Sq^n(xy) = \sum_{i+j=n} Sq^i x \cdot Sq^j y$$

Finally, the Steenrod squares are stable cohomology operations, but they behave ... unstably on the cohomology of a space. Large Steenrod squares act trivially on low dimensional cohomology classes and the last non-trivial operation is the square cup product, more precisely, if $x \in H^n(X; \mathbb{F}_2)$, then

$$Sq^n x = x^2 \quad \text{and} \quad Sq^k x = 0 \text{ when } k > n$$

The next proposition tells us that as a graded vector space the cohomology of any space is a module over the Steenrod algebra on which the Steenrod squares act unstably. One usually writes $\mathcal{U}$ for the category of unstable modules. A graded $\mathbb{F}_2$-algebra which comes with an action of the Steenrod algebra is an $\mathcal{A}_2$-module. When this action is unstable and the product behaves well with the Sq^i (the Cartan formula holds), then we say that it forms an unstable algebra over the Steenrod algebra. We write $\mathcal{K}$ for the category of unstable algebras. A good reference for this categorical point of view is Lionel Schwartz's book [**21**]. We will come back to this in the final section.

Proposition 2.5. *Let X be a space. The mod 2 cohomology $H^*(X, \mathbb{F}_2)$ is a graded $\mathbb{F}_2$-vector space equipped with an unstable action of $\mathcal{A}_2$. Moreover the cup product makes it an unstable algebra, i.e. an unstable module where the Cartan formula holds.*

Remark 2.6. Taken from Wikipedia and a biographical note by George Whitehead. Norman Earl Steenrod (1910 - 1971) received his master's degree from Harvard University in 1934, but before that had problems finishing his studies, probably for financial reasons. He withdrew at some point from college when all his grades were A's. He had to find a job, but on the side wrote a couple of mathematical papers, which finally got him fellowships from where he wanted (he chose Harvard). He then completed his PhD under the direction of Solomon Lefschetz at Princeton University.

Steenrod held positions at the University of Chicago from 1939 to 1942, and the University of Michigan from 1942 to 1947. He moved to Princeton University in 1947, and remained there for the rest of his career. He was editor of the Annals of Mathematics.

In collaboration with Samuel Eilenberg, he was a founder of the axiomatic approach to homology theory (think about the Eilenberg–Steenrod axioms). His impact on algebraic topology is remarkable and the Steenrod algebra has proven to be essential in solving many problems in homotopy theory, from the Hopf invariant one problem, the Adams spectral sequence to Lannes' T-functor.

It can also be measured by looking at a few of the PhD students he had: José Adem (the Adem relations), Peter J. Freyd (the adjoint functor Theorem), Samuel

Gitler (the Brown-Gitler spectra), William S. Massey (the Massey product), Edwin Spanier (the Spanier-Whitehead duality, and his fantastic book, [**23**]), George W. Whitehead (the same as the previous one, but for [**30**]).

3. Sample applications of the Steenrod algebra

The most elementary consequence of the extra structure provided by the action of the Steenrod squares on the mod 2 cohomology of spaces is that one can distinguish homotopy types beyond the range of what one could do with the cup product.

Proposition 3.1. *The spaces $S^2 \vee S^3$ and $\Sigma\mathbb{R}P^2$ are not homotopic. The spaces $S^3 \vee S^5$ and $\Sigma\mathbb{C}P^2$ are not homotopic.*

Proof. The mod 2 cohomology rings of both pairs are isomorphic, there is not a single non-trivial cup product. However the Bockstein distinguishes the first two spaces and there is a Sq^2 in the cohomology of $\mathbb{C}P^2$ by unstability $Sq^2 u = u^2$, hence also in the cohomology of $\Sigma\mathbb{C}P^2$ by stability of Sq^2. □

We have seen that the existence of non necessarily associative real division algebras is closely related to the construction of projective planes. As CW-complexes they are built from a sphere S^n by attaching a $2n$-dimensional cell by an attaching map $f\colon S^{2n-1} \to S^n$. Let us call this space $X = S^n \cup_f e^{2n}$. In integral cohomology the cup square x^2 of a generator $x \in H^n(X;\mathbb{Z})$ is a generator in degree $2n$.

Proposition 3.2. *If x^2 is a generator in $H^{2n}(X;\mathbb{Z})$ then n is a power of 2.*

Proof. If x denotes also the reduction in mod 2 cohomology, then $Sq^n x = x^2$ is a generator in $H^{2n}(X, \mathbb{F}_2)$. But the Adem relations tell us that Sq^n is indecomposable if and only if n is a power of 2. Hence, if n is not a power of two, then $Sq^n x$ factors through some zero cohomology group. For example $Sq^6 = Sq^2 Sq^4 + Sq^5 Sq^1$. Since $Sq^4 x$ lives in $H^{10}(X;\mathbb{F}_2) = 0$ and $Sq^1 x \in H^7(X;\mathbb{F}_2) = 0$, we conclude that $Sq^6 x = 0$. □

In fact Adams proved that n must be equal to $1, 2, 4$ or 8, corresponding to the known real division algebras. His original approach was to use *secondary* cohomology operations, i.e. relations between relations. Even if Sq^{16} is indecomposable, there is a higher relation allowing him to perform an analogous argument as the one presented above.

Another important consequence of the existence of cohomology operations and its precise description is of computational nature. Serre computed the cohomology of Eilenberg-Mac Lane spaces at the prime 2, [**24**]. Let $K(\mathbb{Z}/2, n)$ denote the Eilenberg-Mac Lane space whose homotopy type is determined by the fact that all its homotopy groups vanish except $\pi_n K(\mathbb{Z}/2, n)$, the set of pointed homotopy classes of maps out of the sphere S^n, which is isomorphic to $\mathbb{Z}/2$. Among these we have met $K(\mathbb{Z}/2, 1)$, it is $\mathbb{R}P^\infty$ or BC_2. Serre introduced the notion of *admissible* sequences of integers $I = (i_1, \ldots, i_n)$ where $i_k \geq 2i_{k+1}$. He showed that the cohomology operations $Sq^I = Sq^{i_1} Sq^{i_2} \ldots Sq^{i_n}$ form an $\mathbb{F}_2$-basis of the Steenrod algebra. The *excess* of such an admissible sequence is

$$e(I) = (i_1 - 2i_2) + \cdots + (i_{n-1} - 2i_n) + i_n = i_1 - i_2 - \cdots - i_n$$

Theorem 3.3. *Let $n \geq 1$. The cohomology ring $H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2)$ is a polynomial algebra $\mathbb{F}_2[Sq^I \iota_n]$ where I is admissible with excess $e(I) < n$ and ι_n is the generator in $H^n(K(\mathbb{Z}/2, n); \mathbb{F}_2) \cong \mathbb{F}_2$.*

When $n = 1$ the only sequence of excess zero is the empty sequence, so we see that $H^*(K(\mathbb{Z}/2, 1); \mathbb{F}_2)$ is a polynomial algebra on one generator ι_1, something we already knew, but now we also have the action of the Steenrod algebra.

When $n = 2$ the excess must be zero or one, which corresponds to the operations $Sq^1, Sq^2 Sq^1, Sq^4 Sq^2 Sq^1, \ldots$.

There is a categorical explanation for what is going on here. Given a generator ι_n in degree n there is a free unstable module $F(n)$ generated by ι_n and all possible cohomology operations acting on it in an unstable way. The unstable module $F(0)$ is $\mathbb{F}_2$, and $F(1)$ consists of $\iota_1, Sq^1\iota_1, Sq^2 Sq^1 \iota_1, Sq^4 Sq^2 Sq^1 \iota_1, \ldots$. The free unstable module construction is left adjoint to the forgetful functor from $\mathcal{U}$ to graded $\mathbb{F}_2$-vector spaces. There is also a forgetful functor from $\mathcal{K}$ to $\mathcal{U}$ and to construct from $F(n)$ an unstable algebra one has to throw in all products, but in a way that respects the multiplicative unstability conditions: $Sq^n x = x^2$ when x has degree n. When we look for example at $F(1)$ we see that $Sq^1 \iota_1$ should be ι_1^2, then $Sq^2 Sq^1 \iota_1 = Sq^2 \iota_1^2 = \iota_1^4$, etc. Therefore the free unstable algebra on one generator ι_1 is the free $\mathbb{F}_2$-algebra on $\Sigma F(0)$.

Also, from the point of view of Brown representability, it is not a surprise to have this very strong relationship between the Steenrod algebra and the cohomology of Eilenberg-Mac Lane spaces. Indeed any cohomology theory is representable in the homotopy category of pointed spaces and ordinary cohomology with coefficients in $\mathbb{F}_2$ is precisely represented by the $K(\mathbb{Z}/2, n)$'s, meaning that we have natural isomorphisms

$$H^n(X, \mathbb{F}_2) \cong [X, K(\mathbb{Z}/2, n)]_*$$

for any pointed space X. In particular a natural transformation between H^n and H^{n+k}, i.e. a cohomology operation of degree k, corresponds to the homotopy class of a map $K(\mathbb{Z}/2, n) \to K(\mathbb{Z}/2, n+k)$ and then to a class in $H^{n+k}(K(\mathbb{Z}/2, n); \mathbb{F}_2)$.

To conclude let me try to say a few words about Jean Lannes achievements. Most of the short summary below is taken from [**21**], where precise references to the original research articles can be found. He realized that homological algebra methods in the abelian category of unstable modules $\mathcal{U}$ could be used to study the (non-abelian) category of unstable algebras $\mathcal{K}$. One important and non-trivial fact is that the unstable module $H^*\mathbb{F}_2 = H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$ is an injective object in $\mathcal{U}$. The endofunctor $M \mapsto H^*\mathbb{F}_2 \otimes M$ in the category $\mathcal{U}$ has a left adjoint $T \colon \mathcal{U} \to \mathcal{U}$. The injectivity of $H^*\mathbb{F}_2$ implies that T is an exact functor that commutes with suspension and its left adjoint Ω. One can also perform easy computations such as

$$TF(n) \cong F(n) \oplus F(n-1) \oplus \cdots \oplus F(1) \oplus F(0)$$

What is more interesting is that T commutes with tensor products and then, if K is an unstable algebra, the unstable module TK is actually equipped with a multiplication turning it into an unstable algebra. The same functor T is also left adjoint to tensoring with $H^*\mathbb{F}_2$ in the category $\mathcal{K}$!

The T-functor technology provides access to understanding maps out of $\mathbb{R}P^\infty$, a very difficult problem a priori since the source is CW-complex of infinite dimension. Lannes proved for example that if X is simply connected and $H^*(X; \mathbb{F}_2)$ has finite type (in each degree the cohomology is a finite dimensional $\mathbb{F}_2$-vector space), then the map induced by the cohomology functor

$$[\mathbb{R}P^\infty, X] \to \mathrm{Hom}_{\mathcal{K}}(H^*(X;\mathbb{F}_2), H^*\mathbb{F}_2)$$

is a bijection. A key ingredient is Serre's computation of the cohomolgy of Eilenberg-Mac Lane spaces. In fact there is a much richer structure behind this. One can view the set of homotopy classes of maps as the set of path connected components of the *space* of all maps, $map(\mathbb{R}P^\infty, X)$ endowed with a standard topology. Lannes's T functor allows us to compute the cohomology of the whole space, under very reasonable conditions.

Theorem 3.4. *There is an isomorphism* $TH^*(X, \mathbb{F}_2) \cong H^*(map(\mathbb{R}P^\infty, X); \mathbb{F}_2)$ *for any connected space* X *of finite type with* $\pi_1 X$ *a finite p-group, and* $TH^*(X;\mathbb{F}_2)$ *is also of finite type and trivial in degree* 1.

There is an obvious comparison map since T is left adjoint to tensoring with $H^*\mathbb{F}_2$. It should be adjoint to a map $H^*(X, \mathbb{F}_2) \to H^*\mathbb{F}_2 \otimes H^*(map(\mathbb{R}P^\infty, X); \mathbb{F}_2)$. On the right hand side we see a tensor product of mod cohomology algebras. The Künneth formula tells us that this is the same as $H^*(\mathbb{R}P^\infty \times map(\mathbb{R}P^\infty, X); \mathbb{F}_2)$. Now we have a natural candidate, it is the map induced in mod 2 cohomology by the evaluation map $\mathbb{R}P^\infty \times map(\mathbb{R}P^\infty, X) \to X$.

A celebrated consequence of this theorem is a proof of Sullivan's conjecture, first established by Miller in 1984. Sullivan had the intuition that there were basically no maps from $\mathbb{R}P^\infty$ to a finite complex. Now, for a finite unstable module M it is easy to compute $TM \cong M$ by induction because $T\Sigma^k\mathbb{F}_2 = \Sigma^k\mathbb{F}_2$ (and using exactness). This means that when X is a finite CW-complex (under some mild conditions, finite type, etc.) then $TH^*(X;\mathbb{F}_2) \cong H^*(X;\mathbb{F}_2)$ so that the mapping space $map(\mathbb{R}P^\infty, X)$ has the homotopy type of X, up to 2-completion (what mod 2 cohomology can see). But already all constant maps form a subspace of the mapping space which is homeomorphic to X, hence this mapping space contains only constant maps:

$$ev\colon map(\mathbb{R}P^\infty, X) \to X$$

is a homotopy equivalence.

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