◇ Exercice 1. The Hurewicz homomorphism.

Let (X, x_0) be a pointed and path-connected space. In this exercise you will construct a map

$$\text{Hu}\colon \pi_1(X) \to H_1(X;\mathbb{Z}).$$

1. Explain that a loop ω in X, based at x_0, can be lifted to a map $\tilde{\omega}\colon \Delta^1 \approx I \to X$ and check that $\tilde{\omega}$ is a cycle in $C_1^{sing}(X;\mathbb{Z})$.
2. Show that $\text{Hu}([\omega]) := [\tilde{\omega}]$ is well-defined in $H_1(X;\mathbb{Z})$.
3. Show that Hu is a homomorphism.
4. Check that Hu is a natural transformation of functors, which you explicit.

◇ Exercice 2. The Hurewicz Theorem.

We keep the notations of Exercice 1. The goal is to show that Hu induces an isomorphism

$$(\pi_1 X)_{ab} \cong H_1(X;\mathbb{Z}).$$

1. Let α be a path in X from x_0 to x and γ a loop based at x. Show that the loops $\alpha \star \gamma \star \overline{\alpha}$ and γ induce the same class in $H_1(X;\mathbb{Z})$ via Hu.
2. Show that Hu is surjective.

Remark : Building on these two steps, we could prove the statement for all spaces (not only $CW-$complexes), see here for example. For our purpose, we just prove it for CW-complex using the following steps :

3. Prove that Hu induces an isomorphism as desired when X is a wedge of circles.
4. Prove that Hu induces an isomorphism as desired when X is a 2-dimensional CW-complex.
5. Prove that the general statement follows from the case of a 2-dimensional CW-complex.

◇ Exercice 3. The deficiency of a group.

Let G be a finitely presented group. The deficiency $\mathsf{def}\ G \in \mathbb{Z}$ is the largest difference $|S| - |R|$ between the number of generators S and that of relators R in a presentation of the group G. The aim of this exercise is to show that the deficiency is bounded by the difference of ranks of the first and second homology groups of G.

1. Let A be a finitely generated abelian group. Recall the definition of the integral rank $\mathsf{rank}_{\mathbb{Z}}(A)$ and show that it is equal to the rational rank of the $\mathbb{Q}$-vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$.
   Hint : Recall the fundamental theorem of finitely generated abelian groups.
2. Given a finite presentation $\langle S \mid R \rangle$ of G, let N be the normal subgroup in $F(S)$ generated by R. Show that $\mathsf{rank}_{\mathbb{Z}}\left((H_1(N;\mathbb{Z})_G\right) \leq |R|$.
3. Prove then that $|S| - |R| \leq \mathsf{rank}_{\mathbb{Z}} H_1(G;\mathbb{Z}) - \mathsf{rank}_{\mathbb{Z}} H_2(G;\mathbb{Z})$.
4. Conclude that for any finitely presented group G we have $\mathsf{def}\ G \leq \mathsf{rank}_{\mathbb{Z}} H_1(G;\mathbb{Z}) - \mathsf{rank}_{\mathbb{Z}} H_2(G;\mathbb{Z})$.

$\lozenge$ **Exercice 4. Whitehead's Theorem for $K(G,1)$'s.** We will see here that under certain circumstances the homotopy pushout of $K(\pi,1)$'s is again a $K(\pi,1)$. We consider two injective group homomorphisms $\varphi_1 \colon H \hookrightarrow G_1$ and $\varphi_2 \colon H \hookrightarrow G_2$. We will assume that the normal form theorem for amalgamted sums of groups is known, in particular G_1 and G_2 can be seen as subgroups of $G_1 *_H G_2$ via the canonical maps.

1. Let X' be a connected sub-CW-complex of the connected CW-complex X such that $\pi_1 X' \to \pi_1 X$ is injective. If $p \colon \widetilde{X} \to X$ is the universal cover, show that each connected component of $p^{-1}(X')$ is a universal cover of X'.
2. Give an example of the previous situation where $p^{-1}(X')$ is connected, but not equal to $\widetilde{X}$, and one when this preimage is not connected.
3. Give a counter-example when the map on fundamental groups is not injective.
4. Prove that there exist Eilenberg-Mac Lane spaces $K(H,1)$, $K(G_1,1)$, $K(G_2,1)$ and maps $f_1 \colon K(H,1) \to K(G_1,1)$ and $f_2 \colon K(H,1) \to K(G_2,1)$ inducing φ_1 and φ_2 on π_1.
5. Prove the Whitehead Theorem : The homotopy pushout of f_1 and f_2 is an Eilenberg-Mac Lane space $K(G_1 *_H G_2, 1)$.

Exercice 5. Free action on even spheres.
Prove that the only non-trivial group acting freely on an even dimensional sphere is C_2.
Hint. Use the degree to obtain a homomorphism from the group to C_2.

Relation to group homology. One is interested in free actions on spheres because one could use them to start building a free $\mathbb{Z}G$-resolution. This small result shows one should look for odd dimensional spheres.

$\lozenge$ indicates the weekly assignments.