

◆ **Exercice 1. Universal coefficients.** The objective is to (re)prove the algebraic universal coefficients Theorem and some consequences. Let R be a PID, S a ring, and $\alpha: R \rightarrow S$ a ring map turning S into a (left) R -module.

1. Let C_\bullet be a chain complex of free R -modules. Show that for each integer $n \geq 0$ there is a short exact sequence of S -modules and S -module maps :

$$0 \rightarrow H_n(C_\bullet) \otimes_R S \rightarrow H_n(C_\bullet \otimes_R S) \rightarrow \text{Tor}_R(H_{n-1}(C_\bullet), S) \rightarrow 0$$

The objective is to prove it without using the algebraic Künneth Theorem, but of course you can find inspiration in its proof.

2. When $R = \mathbb{Z}$ deduce from the above result the topological universal coefficients Theorem for singular homology with coefficients in a ring S .
3. Show that $H_n(X; \mathbb{Q}) \cong H_n(X; \mathbb{Z}) \otimes \mathbb{Q}$ for any integer $n \geq 0$.
4. We admit that the short exact sequences from parts 1 and 2 split. Show that the splitting cannot be natural by choosing for example the map $f: \mathbb{R}P^2 \rightarrow S^2$ that collapses the 1-skeleton to a point.

◆ **Exercice 2. Künneth formula for group homology.** Let H and G be two groups. We will see how to adapt the proof of the Künneth Theorem to the case of group rings, which are not PID's in general. Let F_\bullet be a free $\mathbb{Z}G$ -resolution of \mathbb{Z} and P_\bullet be a free $\mathbb{Z}H$ -resolution of \mathbb{Z} . Our goal is to compute $H_*(G \times H; \mathbb{Z})$.

1. Explain why $\mathbb{Z}G$ is not a PID, show in a simple example (e.g. $G = C_2$) that submodules of free modules are not free in general and that $\text{Tor}_i^{\mathbb{Z}G}$ does not vanish for $i \geq 2$.
2. Find two chain complexes C_\bullet and D_\bullet , the first one consisting in free $\mathbb{Z}C_2$ -modules, such that the Künneth Theorem fails for $C_\bullet \otimes_{\mathbb{Z}C_2} D_\bullet$. You could for example look for two (very) different complexes with homology concentrated in degree zero $H_0(C_\bullet) \cong H_0(D_\bullet) \cong \mathbb{Z}$.
3. With F_\bullet and P_\bullet as above, use the Künneth Theorem to (re)prove that $F_\bullet \otimes_{\mathbb{Z}} P_\bullet$ is a free $\mathbb{Z}(G \times H)$ -resolution of \mathbb{Z} .
4. Identify $(F_\bullet \otimes_{\mathbb{Z}} P_\bullet) \otimes_{\mathbb{Z}(G \times H)} \mathbb{Z}$ with $(F_\bullet \otimes_{\mathbb{Z}G} \mathbb{Z}) \otimes_{\mathbb{Z}} (P_\bullet \otimes_{\mathbb{Z}H} \mathbb{Z})$.
5. Establish a Künneth Formula for $H_n(G \times H; \mathbb{Z})$.
6. Apply the formula to the case $G = H = C_2$ (at least in low degrees, give a general formula if you can!).

◆ indicates the weekly assignments. Each exercise is designed for a 25 minute long presentation by a group of two.