EPFL - Autumn semester 2025-2026	J. Scherer
Cohomology rings	**Sheet 5**
Mathematics	6 October 2025

♢ **Exercice 1. Functoriality.** Consider the inclusion $i\colon C_2 \hookrightarrow C_4$ and the projection $p\colon C_4 \to C_2$. Let $F_\bullet$ be the free periodic $\mathbb{Z}C_2$-resolution of $\mathbb{Z}$ and $G_\bullet$ the free periodic $\mathbb{Z}C_4$-resolution.

1. Extend the identity on $\mathbb{Z}$ to a map $\tau\colon F_\bullet \to G_\bullet$ of $\mathbb{Z}C_2$-chain complexes.
2. Compute the induced map $H_n(C_2;\mathbb{Z}) \to H_n(C_4;\mathbb{Z})$ for all $n \geq 0$.
3. Compute the induced map $H_n(C_2;\mathbb{F}_2) \to H_n(C_4;\mathbb{F}_2)$ for all $n \geq 0$.
4. Extend the identity on $\mathbb{Z}$ to a map $\tau\colon G_\bullet \to F_\bullet$ of $\mathbb{Z}C_4$-chain complexes.
5. Compute the induced map $H_n(C_4;\mathbb{Z}) \to H_n(C_2;\mathbb{Z})$ for all $n \geq 0$.
6. Compute the induced map $H_n(C_4;\mathbb{F}_2) \to H_n(C_2;\mathbb{F}_2)$ for all $n \geq 0$.

♢ **Exercice 2. Coefficients modules.** Let G be a group and $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ a short exact sequence of $\mathbb{Z}G$-modules.

1. Prove that there is a connecting homomorphism $\partial\colon H_{n+1}(G;M'') \to H_n(G;M')$ such that there is a long exact sequence in homology
$$\dots H_{n+1}(G;M'') \xrightarrow{\partial} H_n(G;M') \xrightarrow{i_*} H_n(G;M) \xrightarrow{p_*} H_n(G;M'') \xrightarrow{\partial} \dots$$
2. Prove the analogous statement in cohomology.
3. Compute $H_n(C_2;\mathbb{Z}/k)$ for any n and k (the coefficients have the trivial module structure).
4. Compute the connecting homomorphism associated to the exact sequence $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$ of trivial modules, in homology and cohomology (for the group C_2). This connecting homomorphism is called the *Bockstein homomorphism*.

♢ **Exercice 3. Ext and extensions. Reminder from rings and modules ?** Let R be a ring, and A, B two left R-modules. Let ξ denote an extension of A by B in R-modules, i.e. a short exact sequence $0 \to B \to E \to A \to 0$. We denote by $\Theta(\xi) \in \mathrm{Ext}^1_R(A,B)$ the image of the identity under the connecting homomorphism $\partial\colon \mathrm{Hom}_R(A,A) \to \mathrm{Ext}^1_R(A,B)$ induced by the short exact sequence ξ (in the long "Hom-Ext" exact sequence). The objective of this exercise is to prove that Θ is a bijection between the set of equivalences classes of extensions of A by B and $\mathrm{Ext}^1_R(A,B)$. We also fix a projective cover $P \to A$ and call K its kernel.

1. Prove that $\Theta(\xi) = 0$ if and only if ξ splits (as the trivial extension).
2. Show that any element $x \in \mathrm{Ext}^1_R(A,B)$ is in the image of the connecting homomorphism ∂ induced by the short exact sequence $K \xrightarrow{j} P \to A$ (it is $\partial(\beta)$ for a homomorphism $\beta\colon K \to B$).
3. Let X be the pushout of j and β in R-modules, given by the cokernel of $(j, -\beta)$. Show that there is an extension $0 \to B \to X \to A \to 0$ and prove that Θ sends it to x.
4. Show that another preimage β' of x under ∂, as in 2, gives rise to an equivalent extension $0 \to B \to X' \to A \to 0$.
5. For any extension ξ, show that there is a map of extensions from P to E, which yields an isomorphism between the pushout of the left square and E.
6. Conclude that Θ is injective.

$\Diamond$ **Exercice 4. Functorial properties of extensions.** Let $0 \to A \to E \to G \to 1$ be an extension of G by an abelian group A. We fix homomorphisms $\alpha\colon G' \to G$ and $f\colon A \to A'$ a homomorphism of $\mathbb{Z}G$-modules.

1. Show that the pullback of extensions along α corresponds to $\alpha_*\colon H^2(G; A) \to H^2(G'; \alpha^*(A))$.
2. Show that E acts on A' via the projection $E \to G$, so we can form the semi-direct product $A' \rtimes E$.
3. Let N be the subgroup in $A' \rtimes E$ generated by elements of the form $(-f(a), a)$. Show that N is a normal
4. Show that A' is a subgroup of $(A' \rtimes E)/N$ whose quotient is isomorphic to G.
5. Prove that the induced map of extensions corresponds to the map $f_*\colon H^2(G; A) \to H^2(G; A')$.

$\Diamond$ indicates the weekly assignments.