EPFL - Autumn semester 2025-2026 | J. Scherer
Cohomology rings | **Sheet 13**
MATHEMATICS | 8 December 2025

♦ **Exercise 1. On cup products.**

1. Show that all cup products in the cohomology of a suspension are zero (except those with an element of degree zero) by comparing the absolute cup product in $H^*(\Sigma X;\mathbb{Z})$ with a relative one.
2. Prove more generally that iterated cup products of length $\geq n$ vanish if X admits a cover by n contractible subspaces $A_1,\dots,A_n$.
3. Compute $H^*(\Sigma\mathbb{R}P^n;\mathbb{F}_2)$ and $H^*(S^2\vee\cdots\vee S^{n+1};\mathbb{F}_2)$ as graded $\mathbb{F}_2$-algebras.
4. Show that the two spaces above are not homotopy equivalent when $n\geq 2$ in three different ways (use the fundamental group, cohomology with integral coefficients, and the Bockstein homomorphism).
5. Can you do the same for $\Sigma\mathbb{C}P^n$ and $S^3\vee\cdots\vee S^{2n+1}$?
6. Compute the integral cohomology ring structure $H^*(\mathbb{R}P^n;\mathbb{Z})$ by comparing it with the mod 2 computation.

♦ **Exercise 2. The Bockstein at an odd prime.** Let p be an odd prime and $\mathbb{Z}/p\to\mathbb{Z}/p^2\to\mathbb{Z}/p$ a short exact sequence.

1. Define the mod p Bockstein homomorphism β.
2. Compute β for the Moore spaces $M(\mathbb{Z}/p,n)=S^n\cup_p e^{n+1}$ and $M(\mathbb{Z}/p^2,n)=S^n\cup_{p^2}e^{n+1}$.
3. Show that $\beta^2=0$.
4. At the prime 2. Show that the Bockstein is a stable cohomology operation in the sense that it is compatible with the suspension isomorphism $\sigma\colon H^n(X;\mathbb{F}_p)\cong H^{n+1}(\Sigma X;\mathbb{F}_p)$, i.e. $\beta(\sigma(x))=\sigma(\beta(x))$.

♦ indicates the weekly assignments. Each exercise is designed for a 25 minutes presentation by a group of two.