EPFL - Autumn semester 2025-2026	J. Scherer
Cohomology rings	**Sheet 12**
MATHEMATICS	1 December 2025

♦ **Exercise 1. Complex projective spaces.** The complex projective space $\mathbb{C}P^n$ is the quotient of the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ by the relation $x \sim e^{it} \cdot x$ for any $x \in S^{2n+1}$ and any complex number e^{it} of module one. Recall that $\mathbb{C}P^n$ contains $\mathbb{C}P^{n-1}$ as a sub-CW-complex and the attaching map $S^{2n-1} \xrightarrow{q} \mathbb{C}P^{n-1}$ of the top $2n$-dimensional cell is the quotient map. In this exercise you will compute all cup products in $H^*(\mathbb{C}P^n; \mathbb{Z})$.

1. Initialize the induction. Compute the $\mathbb{Z}$-algebra structure of $H^*(\mathbb{C}P^1; \mathbb{Z})$.
2. The inductive step. We assume from now on that $H^*(\mathbb{C}P^{n-1}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}[u]/(u^n)$ where u is a generator of degree two. Explain why it is sufficient to compute $u^k \cup u^\ell$ in $H^*(\mathbb{C}P^n; \mathbb{Z})$ for all $1 \leq k, \ell < n$ with $k + \ell = n$.
3. Show next that this cup product is equivalently computed by a relative cup product for pairs $(\mathbb{C}P^n, \mathbb{C}P^n \setminus P^m)$ where the P^m's are well chosen subspaces (which are homeomorphic to $\mathbb{C}P^m$ for $m = k, \ell$).
4. Use excision to identify the target of the above cup product with $H^{2n}(\mathbb{C}^n, \mathbb{C}^n \setminus 0)$.
5. Show that $\mathbb{C}P^n \setminus P^k$ deformation retracts onto $\mathbb{C}P^{\ell-1}$.
6. Conclude from the previous points and cellular homology that we have isomorphisms :

$$H^{2\ell}(\mathbb{C}P^n, \mathbb{C}P^n \setminus P^k; \mathbb{Z}) \cong H^{2\ell}(\mathbb{C}P^\ell, \mathbb{C}P^{\ell-1}; \mathbb{Z}) \cong H^{2\ell}(\mathbb{C}P^\ell, \mathbb{C}P^\ell \setminus e_\ell; \mathbb{Z}) \cong H^{2\ell}(\mathbb{C}^\ell, \mathbb{C}^\ell \setminus 0)$$

7. Show that the equivalent cup product on pairs $(\mathbb{C}^n, \mathbb{C}^n \setminus \mathbb{C}^m)$ is an isomorphism and conclude.

♦ **Exercise 2. Real division algebras.** Let D be a finite dimensional non-zero division algebra over $\mathbb{R}$. This means that D is a non necessarily associative $\mathbb{R}$-algebra such that every element can be divided by any non-zero element (on the left or on the right). We will show in this exercise that the dimension of D is a power of 2. Let S^{n-1} be the unit sphere in D (seen as $\mathbb{R}^n$) and define $g\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ by $g(x, y) = xy/ \mid xy \mid$.

1. Show that there exist real division algebras of dimension 1, 2 (and 4 and 8 if you know them).
2. Let us write an integer $n \geq 3$ as a sum of powers of 2, so $n = n_1 + n_2 + \cdots + n_k$ with

$$n_1 = 2^{m_1} < n_2 = 2^{m_2} < \cdots < n_k = 2^{m_k}$$

   In $\mathbb{F}_2[x]$ compute $(1 + x)^n$ and show it is equal to $(1 + x^{n_1})(1 + x^{n_2}) \ldots (1 + x^{n_k})$.
3. Show that the resulting polynomial has exactly 2^k terms, so $(1 + x)^n = 1 + x^n$ if and only if n is a power of 2.
4. Deduce from the previous point that the binomial coefficients $\binom{n}{k}$ are all zero mod 2 for $0 < k < n$ if and only if n is a power of 2.
5. Prove that there is an $\mathbb{F}_2$-algebra isomorphism $H^*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}; \mathbb{F}_2) \cong \mathbb{F}_2[a, b]/(a^n, b^n)$ where a and b are pullbacks of $u \in H^1(\mathbb{R}P^{n-1}; \mathbb{F}_2)$ along the two projections.
6. Show that $(a + b)^n = 0$ in $H^*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}; \mathbb{F}_2)$ if and only if n is a power of 2.
7. Prove that g induces a map $\overline{g}\colon \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$.

8. Show that $\overline{g}$ takes a loop ω generating $\pi_1\mathbb{R}P^{n-1}$ in either factors of the product to a non-trivial loop.
9. Show that $\overline{g}$ induces the sum $u \mapsto a + b$ on the first cohomology group.
10. Conclude that n must be a power ot two.

⧫ indicates the weekly assignments. Each exercise is designed for a 25 minutes presentation by a group of two.