EPFL - Autumn semester 2025-2026	J. Scherer
Cohomology rings	**Sheet 11**
MATHEMATICS	24 November 2025

♦ **Exercice 1. Cap product.** There is another important product mixing homology and cohomology. The objective of this exercise is to define this product and study its basic properties. You might encounter this cap product if you study Poincaré duality. Let R be a ring and $n = p + q$.

1. Let $C_\bullet$ be a chain complex of R-modules and $D_\bullet$ be a chain complex of free R-modules. Define
$$\omega\colon \operatorname{Hom}_R(C_q, R) \otimes_R C_q \otimes_R D_p \to D_p$$
by sending $\varphi \otimes c \otimes d$ to $\varphi(c) \cdot d$. For any p-cochain $\psi\colon D_p \to R$ show that $\psi[\omega(\varphi \otimes c \otimes d)] = \nu(\varphi \otimes \psi)(c \otimes d)$.
2. For any $(p-1)$-cochain γ compute $\nu(\varphi \otimes d^p\gamma)(c \otimes d)$ (use a formula about the boundary of ν).
3. Deduce a formula for the boundary of $\omega(\varphi \otimes c \otimes d)$.
Hint. Evaluate it on any $(p-1)$-cochain and observe that these evaluations determine the element in D_{p-1}.
4. Show that ω induces a well-defined pairing $H^q(\operatorname{Hom}_R(C_\bullet, R)) \otimes_R H_n(C_\bullet \otimes D_\bullet) \to H_q(D_\bullet)$.
5. Apply this to the case $C_\bullet = D_\bullet = S_\bullet(X, R)$ to get the *cap product*
$$- \cap -\colon H^q(X; R) \otimes_R H_n(X; R) \to H_p(X; R)$$
6. Let $\psi \in H^p(X; R), \varphi \in H^q(X; R), x \in H_n(X; R)$. Show that $\psi(\varphi \cap x) = (\psi \cup \varphi)(x)$.
7. Let $x = \iota_p \times \iota_q \in H_n(S^p \times S^q; \mathbb{Z})$. Show that the cap product with x provides an isomorphism $H^r(S^p \times S^q; \mathbb{Z}) \cong H_{n-r}(S^p \times S^q; \mathbb{Z})$ for all r.

♦ **Exercice 2. The relative cup product.** The aim of this exercise is to define a relative version of the cross and cup products. We will then do one important computation. Let (X, A) and (Y, B) be two pairs. We recall from Week 10, Exercise 1, that $P_\bullet = S_\bullet(X \times B) + S_\bullet(A \times Y)$ is a subcomplex of $S_\bullet(X \times Y)$ and there is a relative Alexander-Whitney map $\overline{AW}\colon S_\bullet(X \times Y)/P_\bullet \to S_\bullet(X, A) \otimes_R S_\bullet(Y, B)$. We define $n = p + q$.

1. When $\{X \times B, A \times Y\}$ is a pair of excisive subspaces of $X \times Y$, construct a relative cross product.
2. Show that the relative cross product agrees with the absolute one when $A = B = \emptyset$.
3. When $X = Y$ and $\{A, B\}$ is a pair of excisive subspaces of X, construct a relative cup product (use the map $\overline{AW}$, but not the relative cross product).
4. Under the assumptions of the previous parts show that $\Delta^*(x \times y) = x \cup y \in H^n(X, A \cup B)$ for $x \in H^p(X, A)$ and $y \in H^q(X, B)$.
5. Let $T^n = (S^1)^n$ be the n-torus and $u_1 \in H^1(S^1; \mathbb{Z})$ a generator. Show that the n-fold cross product of u_1 with itself is a generator of $H^n(T^n; \mathbb{Z})$.
6. Let $\dot{T}^p$ denote the p-torus of which one removed the top cell. Prove that the cross product $H^p(T^p, \dot{T}^p; \mathbb{Z}) \otimes H^p(T^q, \dot{T}^q; \mathbb{Z}) \to H^n(T^n, \dot{T}^n; \mathbb{Z})$ is an isomorphism.
7. Prove that the cross product $H^p(I^p, \partial I^p; \mathbb{Z}) \otimes H^p(I^q, \partial I^q; \mathbb{Z}) \to H^n(I^n, \partial I^n; \mathbb{Z})$ is an isomorphism, where ∂I^p denotes the boundary of the p-dimensional cube.

♦ indicates the weekly assignments. Each exercise is designed for a 25 minutes presentation by a group of two.