

◆ **Exercice 1. A relative Künneth formula.** The objective is to prove a relative version of the Künneth Theorem. Let  $R$  be a PID, and  $(X, A), (Y, B)$  be two pairs of spaces. We denote by  $P_\bullet$  the subcomplex of  $S_\bullet(X \times Y)$  consisting of sums of elements in  $S_\bullet(X \times B)$  and  $S_\bullet(A \times Y)$ .

1. Let  $(X, A)$  and  $(Y, B)$  be two pairs of spaces. Show that the absolute EZ morphisms induce natural relative chain maps of  $R$ -modules

$$\overline{EZ}: S_\bullet(X, A) \otimes_R S_\bullet(Y, B) \rightarrow S_\bullet(X \times Y)/P_\bullet$$

and a homotopy inverse  $\overline{AW}$ .

2. Define a relative cross product taking values in  $H_*(X \times Y, X \times B \cup A \times Y)$ .
3. If the pair of spaces  $X \times B$  and  $A \times Y$  is excisive, state and prove a relative Künneth formula.
4. Let  $A = \{0\} \cup \{1/n \mid n \geq 1\}$  and  $X = I = [0, 1]$ . Compute  $H_n(X, A; \mathbb{Z})$  for all  $n \geq 0$ , and then  $\bigoplus_{p+q=2} H_p(X, A; \mathbb{Z}) \otimes H_q(X, A; \mathbb{Z})$  as well as  $\bigoplus_{p+q=1} \text{Tor}(H_p(X, A; \mathbb{Z}), H_q(X, A; \mathbb{Z}))$ .
5. Explain why each diagonal square with diagonal vertices  $(1/i, 1/i)$  and  $(1/(i+1), 1/(i+1))$  is a retract of  $X \times A \cup A \times X$  and construct a map  $H_1(X \times A \cup A \times X; \mathbb{Z}) \rightarrow \prod_{i=1}^{\infty} \mathbb{Z}$ . One can admit that it is surjective.
6. Show that  $H_1(X \times A \cup A \times X; \mathbb{Z})$  is uncountable.
7. Conclude that the relative Künneth formula cannot hold for  $(X \times X, X \times A \cup A \times X)$ .

◆ **Exercice 2. Two different spaces and a few computations.**

1. Let  $n \geq 1$ . For any path connected space  $Z$  compute  $H_k(Z \times S^n; \mathbb{Z})$  for any  $k \geq 0$ .
2. Let  $1 = \iota_0, \iota_n, \iota_m$  denote the generators of the integral homology of  $S^n \vee S^m$ . Compute the cross products  $\iota_k \times \iota_\ell \in H_{k+\ell}((S^n \vee S^m)^2; \mathbb{Z})$  in the case  $n \neq m$ .
3. Compute the integral homology of  $\mathbb{R}P^2 \times \mathbb{R}P^2$ , identify all classes that are cross products, and compute the effect of the twist  $T: \mathbb{R}P^2 \times \mathbb{R}P^2 \rightarrow \mathbb{R}P^2 \times \mathbb{R}P^2$  in homology.
4. Same exercise with  $H_*(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{F}_2)$ .
5. Let  $X = S^1 \vee S^2 \vee S^3$  and  $Y = S^1 \times S^2$ . We will prove that these two spaces are not homotopy equivalent. We let  $i_k: S^k \rightarrow X$  be the wedge summand inclusions for  $k = 1, 2, 3$  and  $j_k: S^k \rightarrow Y$  be the inclusion onto the first and second component, for  $k = 1, 2$ . Prove that  $X$  and  $Y$  have isomorphic integral homology groups and isomorphic fundamental group.
6. Identify the universal covers  $\tilde{X}$  and  $\tilde{Y}$  and compute their integral homology. Conclude that  $X$  and  $Y$  are not homotopy equivalent. (We will see later that *cup products* can also help us to distinguish these two homotopy types – but not cross products...).

◆ indicates the weekly assignments. Each exercise is designed for a 25 minutes presentation by a group of two.