♦ **Exercice 1. Yoneda product.** Let R be a ring, $n \geq 2$ and consider two R-modules A and B. An n-fold extension ξ is an exact sequence

$$0 \to B \to E_n \to \cdots \to E_1 \to A \to 0$$

We define the *Yoneda composition product* of the n-fold extension ξ of A by B, and the m-fold extension ξ' of B by C to be the $(n+m)$-fold extension of A by C obtained by splicing together ξ and ξ'.

1. Choose a non-trivial (1-fold) extension ξ of $\mathbb{F}_2$ by $\mathbb{F}_2$ in the category of $\mathbb{F}_2C_2$-modules and show that the iterated composition product with itself determines non-trivial n-fold extensions giving $\mathrm{Ext}^*_{\mathbb{F}_2C_2}(\mathbb{F}_2, \mathbb{F}_2)$ the structure of a polynomial algebra $\mathbb{F}_2[\xi]$.
2. What happens with $\mathrm{Ext}^*_{\mathbb{Z}C_2}(\mathbb{Z}, \mathbb{Z})$? Do the necessary adjustments to compute the ring structure.
3. Show that the reduction mod 2 functor $- \otimes_{\mathbb{Z}C_2} \mathbb{F}_2C_2$ is left adjoint to the "restriction" functor $r^* \colon \mathbb{F}_2C_2 - \mathrm{Mod} \to \mathbb{Z}C_2 - \mathrm{Mod}$ given by considering a $\mathbb{F}_2C_2$-module as a $\mathbb{Z}C_2$-module via the reduction mod 2. See also Exercise 3, Sheet 3, for a more general statement.
4. Compute $\mathbb{Z} \otimes_{\mathbb{Z}C_2} \mathbb{F}_2C_2$ and conclude that $\mathrm{Ext}^*_{\mathbb{F}_2C_2}(\mathbb{F}_2, \mathbb{F}_2)$ is isomorphic to $\mathrm{Ext}^*_{\mathbb{Z}C_2}(\mathbb{Z}, \mathbb{F}_2) = H^*(C_2; \mathbb{F}_2)$.

♦ **Exercice 2. The cohomology of cyclic groups.** Let p be an odd prime and C_p be a cyclic group of order p.

1. Define a free action of C_p on S^1 and describe a cellular decomposition of the circle that yields an exact sequence $0 \to \mathbb{Z} \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ of $\mathbb{Z}G$-modules, where F_0 and F_1 are free and both copies of $\mathbb{Z}$ are trivial modules.
2. Compute $H^0(C_p; \mathbb{Z}C_p)$ and prove that $H^n(C_p; \mathbb{Z}C_p) = 0$ for all $n > 0$.
3. Decompose the above exact sequence into two short exact sequences of $\mathbb{Z}C_p$-modules and construct a composition of two connecting homomorphisms

$$\beta \colon \mathbb{Z} \cong H^0(C_p; \mathbb{Z}) \twoheadrightarrow H^1(C_p; IC_p) \cong H^2(C_p; \mathbb{Z})$$

   We call $v = \beta(1)$.
4. Likewise, show that such connecting homomorphisms induce isomorphisms $H^k(C_p; M) \cong H^{k+2}(C_p; M)$ for any $k > 0$ and $M = \mathbb{Z}$ and $M = \mathbb{F}_p$.
5. We will see in class that $\beta(x) = v \cup x$. Prove that $H^*(C_p; \mathbb{F}_p)$ is isomorphic to $E(u) \otimes \mathbb{F}_p[\bar{v}]$, where $E(u) = \mathbb{F}_p[u]/(u^2)$ is an exterior algebra and $\bar{v}$ is the mod p reduction of v.

Exercice 3. Periodic resolutions. Let $0 \to \mathbb{Z} \to F_{2n-1} \xrightarrow{d_{2n-1}} F_{2n-2} \to \dots \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ be an exact sequence of $\mathbb{Z}G$-modules, where all F_i's are free and $\mathbb{Z}$ is the trivial module (this could come for example from a free action of G on S^{2n-1}). Let M be a $\mathbb{Z}G$-module.

1. Prove that tensoring this sequence with M yields an exact sequence.
2. Show that $F_i \otimes M$ with the diagonal action is isomorphic to $F_i \otimes U(M)$ where U forgets the G-action on M. In particular $F_i \otimes M$ is free over $\mathbb{Z}G$ is $U(M)$ is a free $\mathbb{Z}$-module.
3. Decompose the exact sequence from part 1 into $2n$ short exact sequences and show that there is an iterated connecting homomorphism $\beta\colon H^i(G; M) \to H^{i+2n}(G; M)$ which is an isomorphism for any $i > 0$ and a surjection for $i = 0$.

   Extra information. Over a finite group induced and coinduced modules coincide, so $H^n(G; -)$ vanishes in fact for $n > 0$ on any induced module such as $F_i \otimes U(M)$.
4. Let $v = \beta(1)$ be the image of $1 \in H^0(G; \mathbb{Z})$. Show that β is given by the cup product $v \cup -$.
5. Illustrate this with the example of Q_8 from Week 6.

♦ indicates the two oral presentations for 3 November 2025, to be presented by two pairs. Each presentation will last 25 minutes, followed by questions from students and staff.