◆ **Exercice 1. Diagonal approximation.** Let C_3 be a cyclic group of order 3 generated by t. Let $F_\bullet$ be the periodic resolution where the differentials are given by $t-1$ and the norm $1+t+t^2$. We define a map Δ from $F_\bullet$ to $F_\bullet \otimes F_\bullet$ by sending $1 \in \mathbb{Z}C_3 = F_n$ to the sum of the following elements in $F_i \otimes F_j$: we choose $1 \otimes 1$ if i is even, $1 \otimes t$ if i is odd and j is even, and $1 \otimes t + 1 \otimes t^2 + t \otimes t^2$ if i and j are odd.
Let u and v denote both the augmentation map followed by reduction mod 3, but seen as a cochain in $\mathrm{Hom}_{\mathbb{Z}C_3}(F_1, \mathbb{F}_3)$ and $\mathrm{Hom}_{\mathbb{Z}C_3}(F_2, \mathbb{F}_3)$ respectively.

1. Verify that Δ is a chain map (up to degree 2, and more if you really want).
2. Show that u and v are non-trivial cocycles.
3. Show that $u \cup u = 0$.
4. Show that v is a polynomial generator (all iterated cup products are non-trivial).
5. Conclude that $H^*(C_3; \mathbb{F}_3)$ contains copies of $E(u)$ and $\mathbb{F}_3[v]$ as subalgebras, where $E(u) = \mathbb{F}_3[u]/(u^2)$ is an exterior algebra. In fact $H^*(C_3; \mathbb{F}_3) \cong \mathbb{F}_3[v] \otimes E(u)$ (as graded $\mathbb{F}_3$-algebras).

◇ **Exercice 2. The Baer sum.** The objective of this exercise is to upgrade the bijection from Exercise 3, Sheet 5, to an isomorphism of abelian groups. Given two extensions ξ', ξ'' of A by B, the *Baer sum* ξ is constructed by taking first the pullback $E' \times_A E''$ and then the quotient E under the skew diagonal $B \to E' \times_A E''$ sending b to $(-b, b)$.

1. Show that E provides an extension ξ of A by B.
2. Let τ' and τ'' denote the maps from P to E' and E'' as in Exercise 3, and $\beta', \beta'' \colon K \to B$ the induced map on the kernels. They induce a map $\tau \colon P \to E$ and hence a map $K \to B$. Prove that this map is the sum $\beta' + \beta''$
3. Show that $\Theta(\xi) = \partial(\beta') + \partial(\beta'')$.
4. Conclude that Θ is an isomorphism of abelian groups where zero is given by the split extension $B \oplus A$.
5. Determine all equivalence classes of extensions of $\mathbb{Z}C_2$-modules of $\mathbb{Z}$ by $\mathbb{Z}$ and of $\mathbb{Z}$ by $\mathbb{Z}^\sigma$ (coming from the sign representation).

◇ **Exercice 3. n-fold extensions.** Let R be a ring, $n \geq 2$ and consider two R-modules A and B. An n-fold extension ξ is an exact sequence

$$0 \to B \to E_n \to \cdots \to E_1 \to A \to 0$$

We generate an *equivalence relation* on n-fold extensions by identifying ξ and ξ' if there is a morphism of n-fold extensions $E_* \to E'_*$ (making the appropriate ladder commute).
For two n-fold extensions ξ' and ξ'' we define their *sum* by choosing E_n to be the pushout of $E'_n \leftarrow B \to E''_n$, of which we take the quotient under the skew diagonal $\Delta(B)$. We define E_1 to be the pullback of $E'_1 \to A \leftarrow E''_1$ and complete the new extension with $E_i = E'_i \oplus E''_i$ for $n > i > 1$.
Finally, if $F_\bullet$ is a free resolution of A, let K be the kernel of the map $F_{n-1} \to F_{n-2}$. For any n-fold extension ξ there is a map from $F_\bullet$ to ξ extending the identity on A, inducing hence a map $\beta \colon K \to B$.

1. When $n = 2, 3$ verify that the sum defined above is an n-fold extension of A by B.
2. **Dimension shifting.** Prove that $\mathrm{Ext}^1_R(K_{n-2}, B) \cong \mathrm{Ext}^n_R(A, B)$, where K_{n-2} is the kernel of $F_{n-2} \to F_{n-3}$.
3. Define $\partial(\xi)$ to be the image of β under the connecting homomorphism $\mathrm{Hom}_R(K, B) \to \mathrm{Ext}^1_R(K_{n-2}, B) \cong \mathrm{Ext}^n_R(A, B)$. Construct a map Θ from $\mathrm{Ext}^n_R(A, B)$ to the set of n-fold extensions so as to prove that ∂ is surjective.
4. Compute the image $\Theta(0)$ and show it is equivalent to the n-fold extension

$$0 \to A = A \to 0 \to \cdots \to 0 \to B = B \to 0$$

5. Let $f \colon F_{n-1} \to B$ be any homomorphism and let β be the restriction to K. Show that $\Theta(\beta)$ is equivalent to the image of zero. Conclude that ∂ is injective.
6. Prove that Θ upgrades to a homomorphism of abelian groups.

♢ **Exercice 4. The quaternionic group Q_8.** Let Q_8 be the multiplicative subgroup consisting of $\{\pm 1, \pm i, \pm j, \pm k\}$ in the unit sphere $S^3 \subset \mathbb{H}$. We let Q_8 act by multiplication on S^3 and form the regular polytope K called hexadecachoron as the boundary of the convex hull of Q_8. Hence K has the homotopy type of S^3.

1. Show that K is a simplicial complex which has 16 tetrahedra, 32 triangles, 24 edges, and 8 vertices.
2. Let $F_3 \xrightarrow{d_3} F_2 \to F_1 \to F_0 \xrightarrow{\varepsilon} \mathbb{Z}$ be the augmented simplicial chain complex of K. Show that this is a complex of free $\mathbb{Z}Q_8$-modules and compute $H_0(Q_8; \mathbb{Z})$ and $H_1(Q_8; \mathbb{Z})$.
3. Use the Lefschetz fix point Theorem to identify the kernel of d_3 as a (trivial) $\mathbb{Z}Q_8$-module.
4. Construct a periodic free $\mathbb{Z}Q_8$-resolution of $\mathbb{Z}$.
5. Show that there is an iterated connecting homomorphism $H^i(Q_8; M) \to H^{i+4}(Q_8; M)$ which is an isomorphism for any $i \geq 1$ and an epimorphism for $i = 0$.

♢ indicates the weekly assignments. Groups of three are fine for these longer exercises. This concludes the second round of written assignments, and we will start with oral presentations next. Starting with the ♦ decorated exercise 1 on this Sheet, we will then have two weekly assignments to be presented in class the following week. Each exercise will be presented by two students, the allocated time is 20 minutes (max 25), the objective is to explain the main ideas and key steps, we do not have time to go overtime. Following the instructions is part of the exercise.