Cohomology ring : Solution Sheet 6

◆ **Exercice 1. Diagonal approximation.** This exercise was presented in class. There will not be written solutions for exercises presented orally.

♢ **Exercice 2. The Baer sum.** The objective of this exercise is to upgrade the bijection from Exercise 3, Sheet 5, to an isomorphism of abelian groups. Given two extensions ξ', ξ'' of A by B, the *Baer sum* ξ is constructed by taking first the pullback $E' \times_A E''$ and then the quotient E under the skew diagonal $B \to E' \times_A E''$ sending b to $(-b, b)$.

1. Show that E provides an extension ξ of A by B.
2. Let τ' and τ'' denote the maps from P to E' and E'' as in Exercise 3, and $\beta', \beta'' \colon K \to B$ the induced map on the kernels. They induce a map $\tau \colon P \to E$ and hence a map $K \to B$. Prove that this map is the sum $\beta' + \beta''$
3. Show that $\Theta(\xi) = \partial(\beta') + \partial(\beta'')$.
4. Conclude that Θ is an isomorphism of abelian groups where zero is given by the split extension $B \oplus A$.
5. Determine all equivalence classes of extensions of $\mathbb{Z}C_2$-modules of $\mathbb{Z}$ by $\mathbb{Z}$ and of $\mathbb{Z}$ by $\mathbb{Z}^\sigma$ (coming from the sign representation).

Solution 3. (by Thomas and Blaise)

1. We are given two extensions $\xi' : B \xrightarrow{\iota'} E' \xrightarrow{\pi'} A$ and $\xi'' : B \xrightarrow{\iota''} E'' \xrightarrow{\pi''} A$ and construct the R-module E by taking the quotient $q : E' \times_A E'' \to E$ by the skew diagonal $b \in B \mapsto (-\iota'(b), \iota''(b))$.

   To show that E provides an extension of A by B $\xi : B \xrightarrow{\iota} E \xrightarrow{\pi} A$, one first needs to define the maps π and ι. ι is simply defined as the composition $q \circ (\iota', 0)$. For the map π, let us first consider the map $\begin{pmatrix} \pi' \\ 0 \end{pmatrix} : E' \times_A E'' \to A$. Note that $\pi' \circ \iota' = 0$ since ξ' is exact. In particular, we notice that

   $$\begin{pmatrix} \pi' \\ 0 \end{pmatrix} (\iota'(b), -\iota''(b)) = \pi'(\iota'(b)) + 0 = 0,$$

   so that we get a unique map $\pi : E \to A$ which satisfies $\pi([e_1, e_2]) = \pi'(e_1)$, where $[e_1, e_2]$ is the class of (e_1, e_2) in E.

   As such, we get a sequence $\xi : B \xrightarrow{\iota} E \xrightarrow{\pi} A$. We now still need to show it is exact, namely that ι is injective, π is surjective and $\operatorname{Im} \iota = \operatorname{Ker} \pi$.

   To see that ι is injective, consider an element $b \in B$ such that $[\iota'(b), 0] = [0, 0]$, namely there exists an element $b' \in B$ such that $(\iota'(b), 0) = (\iota'(b'), -\iota''(b'))$. Since ι'' is injective, this means that $b' = 0$. Since ι' is injective, we get that $\iota'(b) = \iota'(0) = 0$, so $b = 0$. In particular, ι is injective.

   Since $\begin{pmatrix} \pi' \\ 0 \end{pmatrix}$ is surjective (one can deduce this by noting that $E' \times_A E'' = \{(e_1, e_2) \in E' \times E'' | \pi'(e_1) = \pi''(e_2)\}$ and that both π' and π'' are surjective), and since $\begin{pmatrix} \pi' \\ 0 \end{pmatrix} = \pi \circ q$, π is necessarily surjective (this is generally true for epimorphisms in any category).

   It then remains to show that $\operatorname{Im} \iota = \operatorname{Ker} \pi$. That $\operatorname{Im} \iota \subseteq \operatorname{Ker} \pi$ comes directly from the fact that $\pi' \circ \iota' = 0$. Now let $[e_1, e_2] \in E$ be such that $\pi([e_1, e_2]) = 0$. By definition, this means that $\pi'(e_1) = 0$, and by explicit description of the pullback, this also means that $\pi''(e_2) = 0$. Since both ξ' and ξ'' are short exact sequences, there exist $b_1, b_2 \in B$ such that $\iota'(b_1) = e_1$ and $\iota''(b_2) = e_2$. Note that

   $$[e_1, e_2] = [\iota'(b_1), \iota''(b_2)] = [\iota'(b_1) + \iota'(b_2), \iota''(b_2) - \iota''(b_2)] = [\iota'(b_1 + b_2), 0] = \iota(b_1 + b_2).$$

   Therefore, we have found a preimage of every element in the kernel of π.

Having shown that ι is injective, π is surjective, and $\operatorname{Im}\iota = \operatorname{Ker}\pi$, we can conclude that ξ is a short exact sequence, and E indeed provides an extension of A by B.

2. Let $p : P \to A$ be a projective cover of A and $i : K \to P$ be the kernel of P. Recall the maps τ' and τ'' from P to E' and E'' exist from the fact that $\pi' : E' \to A$ and $\pi'' : E'' \to A$ are surjective, and that P is a projective cover. In particular, $\tau' : P \to E'$ (resp. $\tau'' : P \to E''$) is such that $p = \pi' \circ \tau'$ (resp. $p = \pi'' \circ \tau''$). We therefore get a map $\tau : P \to E$ given by $q \circ (\tau', \tau'')$ (note that the image of (τ', τ'') is indeed in the pullback $E' \times_A E''$, since $\pi' \circ \tau' = p = \pi'' \circ \tau''$). We also get corresponding (unique) kernel maps $\beta' : K \to B$ and $\beta'' : K \to B$ that satisfy respectively $\iota' \circ \beta' = i \circ \tau'$ and $\iota'' \circ \beta'' = i \circ \tau''$. Consider the commutative diagram

$$\begin{tikzcd}
K \arrow[r, hook, "i"] \arrow[d, dashed, "\beta"'] & P \arrow[r, two heads, "p"] \arrow[d, "\tau"] & A \arrow[d, equal] \\
B \arrow[r, hook, "\iota"'] & E \arrow[r, two heads, "\pi"'] & A
\end{tikzcd}$$

Note that, since $\pi \circ \tau \circ i = p \circ i = 0$ and B is the kernel of π, we have that $i \circ \tau$ factors uniquely through B, namely we get a unique map $\beta : K \to B$ such that $\tau \circ i = \iota \circ \beta$. Therefore, we only need to show that $\beta' + \beta''$ satisfies $\tau \circ i = \iota \circ (\beta' + \beta'')$. Let $x \in K$. Note that

$$\begin{aligned}
\iota \circ (\beta' + \beta'')(x) &= [\iota'(\beta'(x) + \beta''(x)), 0] = [\iota' \circ \beta'(x) + \iota' \circ \beta''(x), 0] \\
&\overset{(*)}{=} [\iota' \circ \beta'(x), \iota'' \circ \beta''(x)] = [\tau' \circ i(x), \tau'' \circ i(x)] = \tau \circ i(x),
\end{aligned}$$

where (*) comes from the quotient relation of E. In particular, we conclude that $\tau \circ i = \iota \circ (\beta' + \beta'')$, which is what we wanted.

3. For the sake of conciseness, we will write $[X, Y]$ to denote the abelian group $\operatorname{Hom}_R(X, Y)$.
Recall that the connecting homomorphism $\partial : [A, A] \to \operatorname{Ext}(A, B)$ comes from using the snake lemma on the following diagram :

$$\begin{tikzcd}
 & & {[A,A]} \arrow[d, red, "-\circ p"] \\
{[P,B]} \arrow[r, "\iota\circ -"'] \arrow[d, "-\circ i"] & {[P,E]} \arrow[r, red, "\pi\circ -"'] \arrow[d, red, "-\circ i"] & {[P,A]} \arrow[d, "-\circ i"] \\
{[K,B]} \arrow[r, red, "\iota\circ -"] \arrow[d, red, "\partial'"] & {[K,E]} \arrow[r, "\pi\circ -"] & {[K,A]} \\
\operatorname{Ext}(A,B) & &
\end{tikzcd}$$

where ∂' is the connecting homomorphism arising from the short exact sequence $K \to P \to A$. The red arrows represent here the diagram chase we are following. Let us then find the image of id_A in $\operatorname{Ext}(A, B)$ by explicitly doing the diagram chase. The image in $[P, A]$ by pre-composition is p. By construction, the map $\tau : P \to E$ is such that $\pi \circ \tau = p$. This is therefore a valid preimage in $[P, E]$. Precomposing with i gives us the element $\tau \circ i = \iota \circ (\beta' + \beta'')$ in $[K, E]$. Clearly then, $\beta' + \beta''$ is a preimage of this element in $[K, B]$, whose image in $\operatorname{Ext}(A, B)$ is $\partial'(\beta' + \beta'') = \partial'(\beta') + \partial'(\beta'')$, since ∂' is a group homomorphism. This is precisely what we wanted to show, namely that $\Theta(\xi) = \partial'(\beta') + \partial'(\beta'')$. It is part of the proof of the snake lemma that the choices of preimages via $\pi \circ -$ and $\iota \circ -$ does not affect the final result.

4. We will show that the Baer sum on extensions of A by B equips the set of extensions modulo equivalences with the structure of an abelian group, and conclude that Θ is an isomorphism of abelian groups.

For two short exact sequences $\xi' : B \to E' \to A$ and $\xi'' : B \to E'' \to A$, write $\xi' + \xi''$ for the sequence $\xi : B \to E \to A$ obtained in part 1.

Showing that extensions of A by B modulo equivalence forms an abelian group essential boils down to using Part 3 to show all axioms of a group. In particular, since Θ is such that $\Theta(\xi+\xi') = \Theta(\xi)+\Theta(\xi')$ for any two extensions, and since it is bijective, we get that the Baer sum naturally equips the equivalence classes of extensions with a group structure, where the opposite of an extension ξ is given by $\Theta^{-1}(-\Theta(\xi))$. All that is left to show then is that $A \oplus B$ is the zero element of this group, i.e. that $\Theta(A \oplus B) = 0$. Consider the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "i"'] \arrow[d, dashed] & P \arrow[r, "p"'] \arrow[d, "\begin{pmatrix} p \\ 0 \end{pmatrix}"] & A \arrow[d, equal] \\
B \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & A \oplus B \arrow[r, "(1\ 0)"] & A
\end{tikzcd}$$

Note that the zero morphism from K to B makes this diagram commute, so that $\Theta(A \oplus B) = \partial'(0) = 0$. Thus the Baer sum equips the set of equivalence classes of extensions from A to B with the structure of abelian groups. Since we showed in Part 3 that Θ preserves that structure, and since it is bijective, we can conclude that it is an isomorphism of abelian groups.

5. As we have seen, this is the same as computing the two modules : $\mathrm{Ext}^1_{\mathbb{Z}C_2}(\mathbb{Z}, \mathbb{Z})$ and $\mathrm{Ext}^1_{\mathbb{Z}C_2}(\mathbb{Z}, \mathbb{Z}^\sigma)$. For the first, let's choose a free $\mathbb{Z}C_2$-resolution of $\mathbb{Z}$ such as the periodic one we have seen before.

$$\cdots \longrightarrow \mathbb{Z}C_2 \xrightarrow{\cdot(1+t)} \mathbb{Z}C_2 \xrightarrow{\cdot(1-t)} \mathbb{Z}C_2 \longrightarrow \mathbb{Z} \longrightarrow 0$$

We now apply the $\mathrm{Hom}_{\mathbb{Z}C_2}(-, \mathbb{Z})$ functor to this resolution, keeping in mind that $\mathrm{Hom}_{\mathbb{Z}C_2}(\mathbb{Z}C_2, \mathbb{Z}) \cong \mathbb{Z}$.

We will also use that a map of multiplication by an element of $\mathbb{Z}C_2$ will become a map of multiplication by an element of $\mathbb{Z}$ where we replace t by 1 because t acts by multiplication by 1 on $\mathbb{Z}$.

We get the following cochain :

$$\cdots \longleftarrow \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow \mathrm{Hom}_{\mathbb{Z}C_2}(\mathbb{Z}, \mathbb{Z}) \longleftarrow 0$$

The quotient we are interested in is $\mathrm{Ker}(\cdot 2)/(0) \cong 0$. In particular, the only extension of $\mathbb{Z}$ by $\mathbb{Z}$ (up to equivalence) is is the split extension $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$, where $\mathbb{Z}C_2$ acts trivially on $\mathbb{Z} \oplus \mathbb{Z}$.

The other will be quite similar. We take the same resolution of $\mathbb{Z}$ but this time we apply it $\mathrm{Hom}_{\mathbb{Z}C_2}(-, \mathbb{Z}^\sigma)$. This will transform the maps in the same way as before except this time t acts by multiplication by -1 on $\mathbb{Z}$, so that we need to replace t by -1. This gives us the following cochain :

$$\cdots \longleftarrow \mathbb{Z}^\sigma \xleftarrow{0} \mathbb{Z}^\sigma \xleftarrow{\cdot(-2)} \mathbb{Z}^\sigma \longleftarrow \mathrm{Hom}_{\mathbb{Z}C_2}(\mathbb{Z}, \mathbb{Z}^\sigma) \longleftarrow 0$$

This time the quotient will be $\mathbb{Z}/2\mathbb{Z}$, so we get the split extension $\mathbb{Z} \to \mathbb{Z}^\sigma \oplus \mathbb{Z} \to \mathbb{Z}$ (where $\mathbb{Z}C_2$ acts trivially) as well as $\mathbb{Z}^\sigma \to \mathbb{Z}^\sigma \oplus \mathbb{Z} \to \mathbb{Z}$, where the action of $\mathbb{Z}C_2$ on the middle term is given by $t \cdot (m, n) = (-m + n, n)$. One can check that this indeed maps to 1 via Θ.

◊ **Exercice 3. *n*-fold extensions.** Let R be a ring, $n \geq 2$ and consider two R-modules A and B. An n-fold extension ξ is an exact sequence

$$0 \to B \to E_n \to \cdots \to E_1 \to A \to 0$$

We generate an *equivalence relation* on n-fold extensions by identifying ξ and ξ' if there is a morphism of n-fold extensions $E_* \to E'_*$ (making the appropriate ladder commute).

For two n-fold extensions ξ' and ξ'' we define their *sum* by choosing E_n to be the pushout of $E'_n \leftarrow B \rightarrow E''_n$, of which we take the quotient under the skew diagonal $\Delta(B)$. We define E_1 to be the pullback of $E'_1 \rightarrow A \leftarrow E''_1$ and complete the new extension with $E_i = E'_i \oplus E''_i$ for $n > i > 1$.
Finally, if $F_\bullet$ is a free resolution of A, let K be the kernel of the map $F_{n-1} \rightarrow F_{n-2}$. For any n-fold extension ξ there is a map from $F_\bullet$ to ξ extending the identity on A, inducing hence a map $\beta\colon K \rightarrow B$.

1. When $n = 2, 3$ verify that the sum defined above is an n-fold extension of A by B.
2. **Dimension shifting.** Prove that $\operatorname{Ext}^1_R(K_{n-2}, B) \cong \operatorname{Ext}^n_R(A, B)$, where K_{n-2} is the kernel of $F_{n-2} \rightarrow F_{n-3}$.
3. Define $\partial(\xi)$ to be the image of β under the connecting homomorphism $\operatorname{Hom}_R(K, B) \rightarrow \operatorname{Ext}^1_R(K_{n-2}, B) \cong \operatorname{Ext}^n_R(A, B)$. Construct a map Θ from $\operatorname{Ext}^n_R(A, B)$ to the set of n-fold extensions so as to prove that ∂ is surjective.
4. Compute the image $\Theta(0)$ and show it is equivalent to the n-fold extension
$$0 \rightarrow A = A \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow B = B \rightarrow 0$$
5. Let $f\colon F_{n-1} \rightarrow B$ be any homomorphism and let β be the restriction to K. Show that $\Theta(\beta)$ is equivalent to the image of zero. Conclude that ∂ is injective.
6. Prove that Θ upgrades to a homomorphism of abelian groups.

Solution 3. This solution will be added later.

◊ **Exercice 4. The quaternionic group Q_8.** Let Q_8 be the multiplicative subgroup consisting of $\{\pm 1, \pm i, \pm j, \pm k\}$ in the unit sphere $S^3 \subset \mathbb{H}$. We let Q_8 act by multiplication on S^3 and form the regular polytope K called hexadecachoron as the boundary of the convex hull of Q_8. Hence K has the homotopy type of S^3.

1. Show that K is a simplicial complex which has 16 tetrahedra, 32 triangles, 24 edges, and 8 vertices.
2. Let $F_3 \xrightarrow{d_3} F_2 \rightarrow F_1 \rightarrow F_0 \xrightarrow{\varepsilon} \mathbb{Z}$ be the augmented simplicial chain complex of K. Show that this is a complex of free $\mathbb{Z}Q_8$-modules and compute $H_0(Q_8; \mathbb{Z})$ and $H_1(Q_8; \mathbb{Z})$.
3. Use the Lefschetz fix point Theorem to identify the kernel of d_3 as a (trivial) $\mathbb{Z}Q_8$-module.
4. Construct a periodic free $\mathbb{Z}Q_8$-resolution of $\mathbb{Z}$.
5. Show that there is an iterated connecting homomorphism $H^i(Q_8; M) \rightarrow H^{i+4}(Q_8; M)$ which is an isomorphism for any $i \geq 1$ and an epimorphism for $i = 0$.

Solution 4. (by Berk, Boran and Zsigmond)

1. The space K is a simplicial complex as it is the convex-hull of finitely many points. If we take a pair of antipodal vertices, say $1, -1$, then the line segment between them lies in the convex-hull. Observe that the mid-point of this segment lies in the interior as it is mid-point of also the segments $[i, -i], [j, -j], [k, -k]$ so there is a open ball around it contained in the convex-hull. Hence a face of K cannot contain a pair of antipodal vertices. But observe apart from this constraint every collection of vertices defines a face in some dimension as any segment between vertices that are not antipodal lies on the boundary of convex-hull. To see last assertion take for example $1, i$. Let v be the mid-point of the segment $[1, i]$. Take orthogonal complement of v in $\mathbb{H}$ and translate it to point v. Then we have a three dimensional affine subspace supporting K on one side. Thus $[1, i]$ lies on boundary.

   We clearly have 8 vertices. To compute the number of faces in each dimension let $[x] = \{x, -x\}$. To count the number of edges we choose 2 of the following four classes $\{[1], [i], [j], [k]\}$. We have 6 ways to do so and for each pair, say $[i], [j]$ we have 4 different edges namely $\{i, j\}, \{-i, j\}, \{-i, -j\}, \{i, -j\}$. Thus there are $6 \cdot 4 = 24$ edges. Using the same principle as counting edges we compute number of triangles. We have $\binom{4}{3}$=4 possible decisions for classes and $2^3 = 8$ number of triangles for each class. So number of triangles is $4 \cdot 8 = 32$. Computing tetrahedra, we have single choice for classes, namely all of them, and $2^4 = 16$ different tetrahedra possible.

In general let $\{e_1, ..., e_n\}$ be standard basis of $\mathbb{R}^n$. Let K_n denote the polyhedron defined by the convex-hull of $\{[e_1], ..., [e_n]\}$. Then number of d-dimensional faces of K_n, where d is smaller than n, is given by the formula $\binom{n}{d+1} \cdot 2^{d+1}$.

2. The space K has the homotopy type of S_3, so we know that truncated free resolution is exact at every point and that $\mathrm{Ker}(d_3) = \mathbb{Z}$. This is because we would normally have that $H_0(K, \mathbb{Z}) = H_3(K, \mathbb{Z}) = \mathbb{Z}$ and $H_1(K, \mathbb{Z}) = H_2(K, \mathbb{Z}) = 0$.

   By the previous part, we know that a k-dimensional subcomplex of K is given by choosing k elements out of the set $\{1, i, j, k\}$ and picking a sign ± 1 for each of them.

   We will define the action of Q_8 on F_k by defining how it acts on the subcomplexes and then extending it linearly. We let $g \in Q_8$ act on by multiplying each element by g from the left.

   For example, if $g = i$, then $g \cdot \{i, -j, k\} = \{i^2, -ij, ik\} = \{-1, -k, -j\} = \{-1, -j, -k\}$.

   Let us now show that this action is free. If a complex is given, then the sum of the elements representing its vertices cannot be zero, since from each basis element and its negation, at most one can appear in the complex. (The vertices lie in $\mathbb{H}$, which is an algebra, and addition is understood within this structure.) Therefore, when we act on this complex with a non-identity element g, the sum of the vertices will also be multiplied by this element individually, and hence the whole sum gets multiplied by it. Consequently, no complex can be invariant under the action of any given element g.

   We know from the lecture notes that $H_0(Q_8; \mathbb{Z}) \simeq \mathbb{Z}$. The commutator subgroup of Q_8 is $\{1, -1\}$, so the first homology group, which is the Abelianization of Q_8, is isomorphic to the Klein-4 group.

3. Let $f_g : S^3 \to S^3$ be the multiplication by g map. Let $H(f)_k : H_k(S^3) \to H_k(S^3)$.

   Recall that Lefschetz number of f is

$$\Lambda_f := \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}(H_k(f))$$

   The Lefschetz Fixed Point Theorem states that if f doesn't have any fixed points, then $\Lambda(f) = 0$.

   Since Q_8 acts on S^3 freely, we know that $\Lambda(f_g) = 0$. Moreover, the formula is much simpler since we have only two homology groups isomorphic to the integers, so in fact the trace reduces to the degree. Thus, in our case, the Lefschetz Fixed Point Theorem just says that $\deg(H_3(f_g)) - \deg(H_0(f_g)) = 0$.

   Let $\phi : X \to Y$ be a continuous map between path-connected spaces. Then, the induced map on homology of ϕ on degree zero, $H_0(\phi)$:$H_0(X) \to H_0(Y)$, is the identity map. Thus, $\deg(H_0(f_g)) = 1$ and so $\deg(H_3(f_g)) = 1$.

   But then the action of g on $H_3(S^3) \simeq \mathbb{Z}$ is multiplication by 1, also known as the identity map.

   Since $H_3(S^3)$ is nothing but the kernel of d_3, we have the desired result.

4. Simply notice that

$$\xrightarrow{d_1} F_0 \xrightarrow{\epsilon} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} \mathbb{Z}$$

   gives a free resolution.

5. Since the chain complex is periodic, we get isomorphisms between homology groups for $i \geq 1$. However, for $i = 0$, we only get an epimorphism since for $i = 4$ we look at the kernel of ϵ quotiented out by the image of d_1 but for $i = 0$ we look at the whole group $F_0 \otimes_{\mathbb{Z}[Q_8]} M$ quotiented out by the image.