Error control in scientific modelling (MATH 500, Herbst)

Sheet 9: Hilbert spaces

Exercise 1

In this exercise we will take a first look at one-dimensional *periodic* problems. A periodic problem is characterized by being invariant to some translations. For example the $\sin$ function is periodic with periodicity 2π, i.e.

$$\sin(x) = \sin(x + 2\pi m) \quad \forall m \in \mathbb{Z},$$

This is nothing else than saying that any translation by an integer multiple of 2π keeps the $\sin$ function invariant. More formally, one can use the translation operator $T_{2\pi m}$ to write this as:

$$T_{2\pi m} \sin(x) = \sin(x - 2\pi m) = \sin(x).$$

The main motivation for looking at periodic problems is that one can exploit periodicity to save computational work. Consider a problem in which we want to find a function $f : \mathbb{R} \to \mathbb{R}$, but *a priori* the solution is known to be periodic with periodicity a. As a consequence of said periodicity it is sufficient to determine the values of f for all x of the interval $[-a/2, a/2)$ to uniquely define f on the full real axis.

In this exercise our concern will be to define appropriate spaces for periodic functions and their approximations in 1D, revisiting some of the points we discussed in the lecture.

First we take a closer look at the sine, which one would consider a nice function. However, we obtain the following result:

(a) Show that sine is square-integrable on bounded domains, i.e. that $\sin \in L^2([-L, L])$ for any $L > 0$.

(b) Argue why sine is not square-integrable on unbounded domains, i.e. why $\sin \notin L^2(\mathbb{R})$.

The fact that $\sin \notin L^2(\mathbb{R})$ is notably related to the very fact that it is a periodic function. This is one motivation for introducing the $L^2_{\text{loc}}(\mathbb{R})$ spaces we discussed in the lecture, which are more forgiving on the boundary of the domain.

(c) Keeping in mind that $[-L, L]$ is compact for $L > 0$, show that $\sin \in L^2_{\text{loc}}(\mathbb{R})$.

The goal is now, based on L^2_{loc}, to introduce a function space taking into account the periodicity of problems. In 1D periodicity is equivalent to saying there exists a lattice $\mathcal{R} = a\mathbb{Z}$ with lattice constant a, i.e.

```
      -3a/2         -a/2         +a/2        +3a/2
  ...  |---------|---------|---------|  ...
```

a a a

where the function is identical in each of the lattice cells compared to any of its neighbours, that is to say

$$f(x) = f(x+R) \qquad \forall R \in \mathcal{R}.$$

One such cell, e.g. $\Omega = [-a/2, a/2)$, we call the unit cell of the problem. Note that both lattice and unit cell are not unique, e.g. a unit cell $[0, a)$ would have worked just as well.

Based on lattice and cell, we define the function space

$$L^2_{\text{per}}(\Omega) = \left\{ f \in L^2_{\text{loc}}(\mathbb{R}) \,\middle|\, f \text{ is } \mathcal{R}\text{-periodic} \right\}$$

with inner product

$$\forall f, g \in L^2_{\text{per}}(\Omega): \qquad \langle f, g \rangle_{L^2_{\text{per}}(\Omega)} = \int_\Omega \overline{f(x)} g(x) \, \mathrm{d}x.$$

Since $f \in L^2_{\text{per}}(\Omega)$ implies $f \in L^2(\Omega)$ all elements of L^2_{per} can be Fourier transformed. This admits the series expansion

$$f = \sum_{G \in \mathcal{R}^*} \widehat{f}_G e_G \quad (*)$$

with the reciprocal lattice $\mathcal{R}^* = \frac{2\pi}{a}\mathbb{Z}$, the plane waves

$$e_G(x) = \frac{1}{\sqrt{|\Omega|}} e^{iG \cdot x}$$

and the Fourier coefficients

$$\widehat{f}_G = \int_\Omega f(x) \overline{e_G(x)} \, \mathrm{d}x.$$

(d) Based on the basis expansion $(*)$ argue why $L^2_{\text{per}}(\Omega)$ is separable, by explicitly constructing a countable dense subset. This requires a few steps of reasoning. Follow this path to convince yourself of this result:

- Consider the set V defined by

$$V = \bigcup_{N \in \mathbb{N}} V_N, \qquad V_N = \left\{ \sum_{\substack{G \in \mathcal{R}^*, \\ |G| \le N}} q_G e_G \,\middle|\, q_G = q_G^{\text{Re}} + i q_G^{\text{Im}}, \quad q_G^{\text{Re}}, q_G^{\text{Im}} \in \mathbb{Q} \right\}.$$

- Show the set V to be countable using the standard theorems discussed in the wikipedia article on countable sets.

- Argue that the completion of V under the norm of $L^2_{\text{per}}(\Omega)$ is indeed $L^2_{\text{per}}(\Omega)$, i.e. that V is a dense subset of $L^2_{\text{per}}(\Omega)$. That is, for each $f \in L^2_{\text{per}}(\Omega)$ construct a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in V with

$$\lim_{n\to\infty} \|f_n - f\|_{L^2_{\text{per}}(\Omega)} = 0.$$

- Conclude regarding the separability of $L^2_{\text{per}}(\Omega)$.

Solutions

Thm 1. Any continuous function on a compact set $K \subset \mathbb{R}$ is (Lebesgue-)integrable.

1a)

(*The abstract way*) The interval $[-L, L]$ is compact and function $|\sin(.)|^2$ is continuous on $\mathbb{R}$, thus $\int_{-L}^{L} |\sin(x)|^2 dx < \infty$. Thus $\sin \in L^2([-L, L])$ for any $L > 0$.

(*The concrete way:*) $\int_{-L}^{L} |\sin(x)|^2 dx = [\frac{1}{2}(x - \sin(x)\cos(x))]_{-L}^{L} = L - \sin(L)\cos(L) < \infty.$

1b)

One can verify that $I_L := \int_{-L}^{L} |\sin(x)|^2 dx$ diverges to $+\infty$ as $L \to \infty$.

1c)

As in 1a), continuity of $|\sin(.)|^2$ over all of $\mathbb{R}$ is enough to show using Theorem 1 above that $\sin \in L^2_{\text{loc}}(\mathbb{R})$.

1d)

I. V is countable: $\mathbb{Q}$ is countable. V_N is a finite product of countable sets, thus countable. V is a countable union of countable sets, thus countable.

II. V is dense in $L^2_{\text{per}}(\Omega)$. Let $f \in L^2_{\text{per}}(\Omega)$ and and $\varepsilon > 0$. Since $(*)$ is a basis expansion, there exists an N such that

$$\left\| f - \sum_{\substack{G\in\mathcal{R}^* \\ |G|\le N}} \hat{f}_G e_G \right\|_{L^2_{\text{per}}(\Omega)} \le \frac{\varepsilon}{2}.$$

Furthermore, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists rational complex coefficients q_G such that

$$\left\| \sum_{\substack{G\in\mathcal{R}^* \\ |G|\le N}} \hat{f}_G e_G - \sum_{\substack{G\in\mathcal{R}^* \\ |G|\le N}} q_G e_G \right\|_{L^2_{\text{per}}(\Omega)} \le \frac{\varepsilon}{2}.$$

Finally, combining by the triangle inequality:

$$\left\| f - \sum_{\substack{G \in \mathcal{R}^* \\ |G| \leq N}} q_G e_G \right\|_{L^2_{\mathrm{per}}(\Omega)} \leq \varepsilon.$$

Thus V is dense in $L^2_{\mathrm{per}}(\Omega)$. With V being countable, this verifies that $L^2_{\mathrm{per}}(\Omega)$ is separable.