Error control in scientific modelling (MATH 500, Herbst)

```julia
using PlutoTeachingTools
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Sheet 10: Operators

Exercise 1

We work on the real line $\mathbb{R}$. Consider the following spaces: $C^\infty(\mathbb{R})$, $C_0^\infty(\mathbb{R})$, $C_0^2(\mathbb{R})$, $H^2(\mathbb{R})$, and $L^2(\mathbb{R})$.

(a) Which of these spaces are Hilbert spaces? In each case, explain your answer briefly in one sentence.

(b) Arrange these spaces by inclusion, where possible. Argue whether each inclusion is strict.

Solutions

(a) solution

Out of these, only $H^2(\mathbb{R})$ and $L^2(\mathbb{R})$ are Hilbert spaces, when considered with their respective inner products.

(b) solution

We have $C_0^\infty(\mathbb{R}) \subset C_0^2(\mathbb{R}) \subset H^2(\mathbb{R}) \subset L^2(\mathbb{R})$ and $C_0^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R})$.

All inclusions are strict.

Exercise 2

In this exercise we want to discuss the subtlety between symmetric and self-adjoint operators on one example: the Laplacian on $L^2(\mathbb{R})$. We take the operator as

$$\mathcal{A} = -\Delta = -\frac{d^2}{dx^2}$$

with domain $D(\mathcal{A}) = C_0^\infty(\mathbb{R})$.

As mentioned in the Lecture the Laplacian is self-adjoint if the domain is taken to be $H^2(\mathbb{R})$. However, in this setting $\mathcal{A}$ is *not* self-adjoint exactly because the domain is too small. To see that proceed as follows:

(a) Show that $\mathcal{A}$ is symmetric.

Consider the definition of the adjoint $\mathcal{A}^*$, which is the operator $\mathcal{A}^* : D(\mathcal{A}^*) \to L^2(\mathbb{R})$ defined by $f \mapsto \mathcal{A}^* f$, where $\mathcal{A}^* f \in L^2(\mathbb{R})$ is the unique element satisfying

$$\langle u, \mathcal{A}^* f\rangle = \langle \mathcal{A}u, f\rangle. \qquad \forall u \in C_0^\infty(\mathbb{R}), \forall f \in D(\mathcal{A}^*), \tag{1}$$

where we take the domain $D(\mathcal{A}^*)$ to be as large as possible.

(b) Due to the implicit definition of $\mathcal{A}^*$ and $D(\mathcal{A}^*)$ it is not immediately obvious what exactly this domain should be. However, with thinking back to our definition of weak derivatives one easily identifies (1) to be satisfied for all f from a particular Sobolev space. Which one ?

(c) Based on your results in **(b)** argue why $\mathcal{A}$ cannot be self-adjoint.

Solutions

(a) solution

Let $u, v \in D(\mathcal{A})$. Since $D(\mathcal{A}) = C_0^\infty(\mathbb{R})$, integration by parts can be used to arbitrary order.

$$\begin{aligned}\langle u, \mathcal{A}v\rangle_{L^2(\mathbb{R})} &= \langle u, -\Delta v\rangle_{L^2(\mathbb{R})} \\ &= -\int_{\mathbb{R}} \overline{u(x)} v''(x) dx \\ &= -\int_{\mathbb{R}} \overline{u''(x)} v(x) dx \\ &= \langle \mathcal{A}u, v\rangle_{L^2(\mathbb{R})}.\end{aligned}$$

Thus $\mathcal{A}$ is symmetric.

(b) solution

Consider the Sobolev space $H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \mid Df \in L^2(\mathbb{R}), D^2 f \in L^2(\mathbb{R})\}$.

By definition of the weak derivative, we have for every $u \in C_0^\infty(\mathbb{R})$ and every $f \in H^2(\mathbb{R})$ that

$$\langle D^2 f, u\rangle = \langle f, D^2 u\rangle.$$

Rearanging both sides, we have:

$$\langle u, -D^2 f\rangle = \langle -\Delta u, f\rangle$$

We have thus shown that $H^2(\mathbb{R}) \subseteq D(\mathcal{A}^*)$.

(c) solution

We have $C_0^\infty(\mathbb{R}) \subsetneq H^2(\mathbb{R})$. Combined with the result $H^2(\mathbb{R}) \subseteq D(\mathcal{A}^*)$ from (b), this yields $D(\mathcal{A}) \subsetneq D(\mathcal{A}^*)$. Thus $\mathcal{A}$ is not self-adjoint.

Exercise 3

We stick to our example from the previous exercise, the Laplacian $\mathcal{A} = -\Delta$ on the Hilbert space $L^2(\mathbb{R})$ with domain $D(\mathcal{A}) = C_0^\infty(\mathbb{R})$.

You may think the distinction between symmetric and self-adjoint is subtle and only a mathematical peculiariaty. Howver, it turns out this difference very much has physical significance.

We will discuss later that the negative Laplacian has only real and positive eigenvalues if one choses the usual domain $D(-\Delta) = H^2(\mathbb{R})$. This makes physical sense as this operator measures the kinetic energy for a "free electron", which thus is necessarily a positive quantity. Further since the electron can travel at any positive speed, it is reasonable that $\sigma(-\Delta) = \mathbb{R}$.

However, with $D(\mathcal{A}) = C_0^\infty(\mathbb{R})$ as we choose it here, we get the *unphysical result* that $\sigma(-\Delta) = \mathbb{C}$, the entire complex plane. As we will see now.

(a) To find the spectrum of $\mathcal{A}$ we will construct its resolvent set $\rho(\mathcal{A})$, which is

$$\rho(\mathcal{A}) = \{z \in \mathbb{C} \mid (\mathcal{A} - z) : C_0^\infty(\mathbb{R}) \to L^2(\mathbb{R}) \text{ is invertible with bounded inverse}\}$$

Invertible in particular means that $\mathcal{A} - z$ is bijective. We will show that $\mathcal{A} - z$ is *not* bijective for all $z \in \mathbb{C}$. To arrive at this point proceed as follows:

1. Argue why $\mathcal{A}f \in C_0^\infty(\mathbb{R})$ for all $f \in C_0^\infty(\mathbb{R})$.
2. Show that $\mathcal{A} : C_0^\infty(\mathbb{R}) \to L^2(\mathbb{R})$ is not surjective. (Hint below)
3. From this conclude that $\rho(\mathcal{A}) = \emptyset$ and $\sigma(\mathcal{A}) = \mathbb{C}$.

Hint

In point 2 you need to find a function in $L^2(\mathbb{R})$, which cannot be generated by applying the Laplacian to a function from $C_0^\infty(\mathbb{R})$. For this note the special property of $\mathcal{A}$ discussed in 1.

Solutions

(a) solution

1. $C_0^\infty(\mathbb{R})$ is closed under differentation to any order. Thus $\mathrm{Im}\mathcal{A} \subseteq C_0^\infty(\mathbb{R})$.

2. We have $C_0^\infty(\mathbb{R}) \subsetneq L^2(\mathbb{R})$. Combined with (1), it follows that $\mathcal{A}$ cannot be surjective.

3. For every $z \in \mathbb{C}$, we have the image $\mathrm{Im}\,(\mathcal{A} - z) \subseteq C_0^\infty(\mathbb{R})$. Thus the operator $(A - zI)$ is not surjective either. It follows that the resolvent set is empty, and the spectrum is $\sigma(\mathcal{A}) = \mathbb{C}$.