Exercise Sheet 13
Algebraic Number Theory

January 7, 2026

Exercise 1 (Dirichlet series, analytic properties)**.** Let $(a_n)_n$ be a sequence of complex numbers and consider the series, for $s \in \mathbb{C}$, $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$. This is called Dirichlet series.

1. Let $\sigma_a((a_n)_n) = \inf\{\sigma \in \mathbb{R} \mid \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} < \infty\}$ and suppose that $\sigma_a((a_n)_n) < \infty$. Show that
$$\{s \in \mathbb{C} \mid \operatorname{re}(s) > \sigma_a\} \to \mathbb{C}; s \mapsto \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$
is well-defined.
2. Let $\sigma_c((a_n)_n) = \inf\{\sigma \in \mathbb{R} \mid \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \text{ converges}\}$. Show that, if $\sigma_c((a_n)_n) < \infty$, then
$$\sigma_c((a_n)_n) \leq \sigma_a((a_n)_n) \leq \sigma_c((a_n)_n) + 1.$$
3. Suppose that $\sigma_c((a_n)_n) < \infty$. Recall complex analytic arguments to say that the function
$$L((a_n)_n, \bullet)\colon \{s \mid \operatorname{re}(s) > \sigma_c((a_n)_n)\};\ s \mapsto L((a_n)_n, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$
is holomorphic and its derivative is given, for $\operatorname{re}(s) > \sigma_c((a_n)_n)$, by $L'((a_n)_n, s) = -\sum_{n=1}^{\infty} \frac{(\log n) a_n}{n^s}$.[1]
4. Suppose now that $a_n \geq 0$ for all n. Show that if $\sum_{n=1}^{\infty} \frac{a_n}{n^\sigma}$ converges for some $\sigma > 0$ and $L((a_n)_n, s)$ can be analytically extended in a neighborhood of σ, then there exists $\epsilon > 0$ so that $\sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma-\epsilon}}$ converges.

Solution.

1. We have $|n^s| = n^{\operatorname{re}(s)}$. Hence for s with real part $\operatorname{re}(s) > \sigma_a$ the series is absolutely convergent.
2. Denote $\sigma_c((a_n)_n) = \sigma_c$. Since absolute convergence is stronger than conditional convergence (by triangle inequality) we have $\sigma_a((a_n)_n) \geq \sigma_c$. Suppose $s \in \mathbb{C}$ satisfies $\operatorname{re}(s) > \sigma_c + 1$. Then $\operatorname{re}(s) - 1 > \sigma_c$. Let $\epsilon > 0$ be so that $\operatorname{re}(s) - 1 > \sigma_c + \epsilon$. Then $\frac{a_n}{n^{\operatorname{re}(s)-1-\epsilon}} \to 0$ as $n \to \infty$ and in particular, there is $C > 0$ so that for all n we have $|a_n| \ll C n^{\operatorname{re}(s)-1-\epsilon}$
$$\begin{aligned} \left|\frac{a_n}{n^s}\right| &= \frac{|a_n|}{n^{\operatorname{re}(s)}} \\ &\leq C \frac{n^{\operatorname{re}(s)-1-\epsilon}}{n^{\operatorname{re}(s)}} \leq \frac{C}{n^{1+\epsilon}}. \end{aligned}$$
Hence the series converges absolutely for $s \in \mathbb{C}$ with $\operatorname{re}(s) > 1 + \sigma_c$ and so $1 + \sigma_c \geq \sigma_a$.
3. The argument coming from complex analysis is called Weiestrass theorem and it is the following: Suppose we are given a sequence of holomorphic functions $f_m\colon \Omega \to \mathbb{C}$, where Ω is a non empty open subset of $\mathbb{C}$. Suppose that f_m converges pointwise to a function f and that the convergence, when we restrict each f_m to a compact subset $K \subset \Omega$ is uniform, for any compact $K \subset \Omega$. Then f is holomorphic and $f'_m \to f'$, moreover the convergence is uniform on compact subsets.

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[1] Typo in the Exercise Sheet

To apply this theorem, we need to show that for $\mathrm{re}(s) > \sigma_c((a_n)_n)$ the convergence of the series $L_m(s) = \sum_{n=1}^m \frac{a_n}{n^s} \xrightarrow{m\to\infty} L(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}$ is uniform on compact subsets. Denote $\sigma_c = \sigma_c((a_n)_n)$. Let $K \subset \{\mathrm{re}(s) > \sigma_c\}$ be a compact set and let

$$r = \min_{z\in K} |\mathrm{re}(z) - \sigma_c| > 0.$$

This min exists since K is compact and it is strictly positive.

Fix a reference point $s_0 \in \mathbb{C}$ with $\mathrm{re}(s_0) > \sigma_c$ and $\mathrm{re}(s_0) - \sigma_c < r/2$. We denote

$$L_t(s_0) = \sum_{1\leq n\leq t} \frac{a_n}{n^{s_0}}$$

Since $(L_m(s_0))_m$ converges, there is a $C > 0$ so that for all m one has $|L_m(s_0)| \leq C$. Let $s \in K$. By summation by parts,

$$\begin{aligned} L_M(s) - L_N(s) = \sum_{n=N+1}^{M} \frac{a_n}{n^s} &= \sum_{N<n\leq m} \frac{a_n}{n^{s_0}} \frac{1}{n^{s-s_0}} \\ &= \frac{1}{M^{s-s_0}} \sum_{N<n\leq M} \frac{a_n}{n^{s_0}} + (s-s_0) \int_N^M \left(\sum_{N<n\leq t} \frac{a_n}{n^{s_0}} \right) \frac{1}{t^{s-s_0+1}} \,\mathrm{d}t \\ &= \frac{1}{M^{s-s_0}} \left(L_M(s_0) - L_N(s_0)\right) + (s-s_0) \int_N^M \left(L_t(s_0) - L_N(s_0)\right) \frac{1}{t^{s-s_0+1}} \,\mathrm{d}t. \end{aligned}$$

The first term on the right hand side is bounded

$$\left| \frac{1}{M^{s-s_0}} \left(L_M(s_0) - L_N(s_0)\right) \right| \leq \frac{2C}{M^{\mathrm{re}(s-s_0)}} \leq \frac{2C}{M^{r/2}}$$

uniformly on $s \in K$. The second term on the right hand side is bounded

$$\begin{aligned} \left| (s-s_0) \int_N^M \left(L_t(s_0) - L_M(s_0)\right) \frac{1}{t^{s-s_0+1}} \,\mathrm{d}t \right| &\leq 2C \sup_{z\in K} |z - s_0| \int_N^\infty \frac{1}{t^{r/2+1}} \,\mathrm{d}t \\ &\leq 2C \sup_{z\in K} |z - s_0| N^{-r/2}. \end{aligned}$$

The sup is finite since K is compact. Hence, we see all at once that for all $s \in K$ the sequence $(L_M(s))_M$ is a Cauchy sequence and that the convergence is uniform for all $s \in K$. The statement about derivative follows from Weierstrass and $\frac{\mathrm{d}}{\mathrm{d}s} \frac{1}{n^s} = -\frac{\log n}{n^s}$.

4. Call $L(s) = L((a_n)_n, s)$. Since $L(s)$ is holomorphic around σ there are a $\sigma' > \sigma$ so that the Taylor series of L centered around σ' has convergence radius larger than $\sigma' - \sigma$. Let $\epsilon < \sigma' - \sigma$, then

$$L(\sigma - \epsilon) = \sum_{k=0}^\infty \frac{L^{(k)}(\sigma')}{k!} (\sigma - \sigma' - \epsilon)^k.$$

By the exercise before, iterated in k, we have

$$L^{(k)}(\sigma') = \sum_{n=1}^\infty \frac{(-1)^k (\log n)^k a_n}{n^{\sigma'}}.$$

Hence,

$$L(\sigma - \epsilon) = \sum_{k=0}^\infty \frac{(-1)^k (\sigma - \sigma' - \epsilon)^k}{k!} \sum_{n=1}^\infty \frac{(\log n)^k a_n}{n^{\sigma'}} = \sum_{k=1}^\infty \frac{(\sigma' - \sigma + \epsilon)^k}{k!} \sum_{n=1}^\infty \frac{(\log n)^k a_n}{n^{\sigma'}}.$$

Now the terms in the summation are all positive, hence we can rearrange the sums

$$L(\sigma - \epsilon) = \sum_{n=1}^\infty \frac{a_n}{n^{\sigma'}} \sum_{k=0}^\infty \frac{\left(\log n(\sigma' - \sigma + \epsilon)\right)^k}{k!} = \sum_{n=1}^\infty \frac{a_n}{n^{\sigma'}} e^{\log(n)(\sigma' - \sigma + \epsilon)} = \sum_{n=1}^\infty \frac{a_n}{n^{\sigma - \epsilon}},$$

showing that $\sum_{n=1}^\infty \frac{a_n}{n^{\sigma-\epsilon}}$ is convergent.

Exercise 2. Let $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ be a the n'th cyclotomic extension. *Assume* that $\zeta_{\mathbb{Q}(\zeta_n)}(s) = \zeta(s) \prod_{\substack{\chi \pmod n \\ \chi \neq 1}} L(\chi, s)$, where on the left is the Dedekind zeta function of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ and the product on the right is ranging over all the non-trivial[2] characters in $\widehat{\mathbb{Z}/n\mathbb{Z}^\times}$

Show that for all *non trivial* $\chi \in \widehat{\mathbb{Z}/n\mathbb{Z}^\times}$ we have $L(\chi, 1) \neq 0$.

Solution. We know, from the class number formula, that $\zeta_{\mathbb{Q}(\zeta_n)}$ has a simple pole at $s = 1$ and that $\zeta(s)$ as well. If some of the $L(\chi, 1) = 0$ for χ non-trivial, then the function $\zeta(s) \prod_{\chi \neq 1} L(\chi, s)$ would have a removable singularity at $s = 1$, which is not the case.

We can also argue without appealing to the class number formula. From the last exercise we deduce that a Dirichlet series defined by a non-negative sequence must have a singularity at $s = \sigma_c$, the abscissa of conditional convergence. On the other hand, we see that if $L(\chi, 1) = 0$ for some non-trivial χ, then $\zeta(s) \prod_{\chi \neq 1} L(\chi, s)$ would be holomorphic around $s = 1$ and so $\sigma_c \leq 0$ but this is readily false for $\zeta_{\mathbb{Q}(\zeta_n)}(s)$.

Exercise 3. Merry Christmanas and Happy new year

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[2] Non-trivial was missing in first formulation, I am sorry.