

Exercise Sheet 13

Algebraic Number Theory

December 17, 2025

Exercise 1 (Dirichlet series, analytic properties). Let $(a_n)_n$ be a sequence of complex numbers and consider the series, for $s \in \mathbb{C}$, $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$. This is called Dirichlet series.

1. Let $\sigma_a((a_n)_n) = \inf\{\sigma \in \mathbb{R} \mid \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} < \infty\}$ and suppose that $\sigma_a((a_n)_n) < \infty$. Show that

$$\{s \in \mathbb{C} \mid \operatorname{re}(s) > \sigma_a\} \rightarrow \mathbb{C}; s \mapsto \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is well-defined.

2. Let $\sigma_c((a_n)_n) = \inf\{\sigma \in \mathbb{R} \mid \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \text{ converges}\}$. Show that, if $\sigma_c((a_n)_n) < \infty$, then

$$\sigma_c((a_n)_n) \leq \sigma_a((a_n)_n) \leq \sigma_c((a_n)_n) + 1.$$

3. Suppose that $\sigma_c((a_n)_n) < \infty$. Recall complex analytic arguments to say that the function

$$L((a_n)_n, \bullet): \{s \mid \operatorname{re}(s) > \sigma_c((a_n)_n)\}; s \mapsto L((a_n)_n, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is holomorphic and its derivative is given, for $\operatorname{re}(s) > \sigma_c((a_n)_n)$, by $L'((a_n)_n, s) = \sum_{n=1}^{\infty} \frac{(\log n)a_n}{n^s}$.

4. Suppose now that $a_n \geq 0$ for all n . Show that if $\sum_{n=1}^{\infty} \frac{a_n}{n^\sigma}$ converges for some $\sigma > 0$ and $L((a_n)_n, s)$ can be analytically extended in a neighborhood of σ , then there exists $\epsilon > 0$ so that $\sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma-\epsilon}}$ converges.

Exercise 2. Let $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ be the n 'th cyclotomic extension. Assume that $\zeta_{\mathbb{Q}(\zeta_n)}(s) = \zeta(s) \prod_{\chi \pmod{n}} L(\chi, s)$, where on the left is the Dedekind zeta function of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ and the product on the right is ranging over all the characters in $\widehat{\mathbb{Z}/n\mathbb{Z}^\times}$

Show that for all *non trivial* $\chi \in \widehat{\mathbb{Z}/n\mathbb{Z}^\times}$ we have $L(\chi, 1) \neq 0$.

Exercise 3. Merry Christmas and Happy new year