Exercise Sheet 12
Algebraic Number Theory

January 12, 2026

Exercise 1. Let $K := \mathbb{Q}(\sqrt{-47})$. The aim of this exercise is to determine the class group of K.

1. Show that every ideal of O_K is equivalent to a divisor of (12).

   Hint: Use Corollary 4.1 from the class to show that any ideal is equivalent to an integral ideal containing an integer smaller than 5.

2. Determine the prime factorization of (12).

3. Let
$$\theta := \frac{1+\sqrt{-47}}{2}.$$
   Determine the prime factorization of the ideals (θ) and of $(4+\theta)$.

4. Conclude that
$$\mathrm{Cl}(O_K) \cong \mathbb{Z}/5\mathbb{Z}.$$

Exercise 2. Determine the class group of $\mathbb{Q}(\sqrt{-19})$.

Exercise 3. Determine the class group of $\mathbb{Q}(\sqrt{-5})$.

Exercise 4. Let A be a principal ideal domain, Q its field of fractions, K/Q a separable extension of finite degree d and B the integral closure of A in B. Recall that for any A-submodule $M \subset B$ we define the discriminant
$$\mathrm{disc}(M) = \mathrm{disc}(z_1, \ldots, z_d),$$
where $(z_1, \ldots, z_d)$ is any A-basis of M. Let $M \subset B$ be rank d-submodule of B, show that there are $a_1, \ldots, a_d \in A$ so that
$$\mathrm{disc}(M) = (a_1 \cdots a_d)^2 \mathfrak{D}_{B/A}. \tag{1}$$
In particular show that if $A = \mathbb{Z}$, then
$$\mathrm{disc}(M) = [B : M]^2 \mathfrak{D}_{B/A}, \tag{2}$$

Exercise 5. Consider $K := \mathbb{Q}(\sqrt[3]{7})$.

1. Show that $O_K = \mathbb{Z}[\sqrt[3]{7}]$

   Hint: Compute the discriminant of both O_K and $\mathbb{Z}[\sqrt[3]{7}]$ and deduce that the only possible prime divisors of $[O_K : \mathbb{Z}[\sqrt[3]{7}]]$ are 3 and 7, conclude by contradiction.

2. Find the signature (r_1, r_2) of K.

3. Prove that $\mathrm{Cl}(O_K) \simeq \mathbb{Z}/3\mathbb{Z}$.