Class Number Formula

$$r_K(m) = \left| \{ \mathfrak{a} \subset \mathcal{O}_K \quad \mathrm{Nr}(\mathfrak{a}) = m \} \right|$$

THEOREM 4.10. *[The class number formula] As $X \to +\infty$ we have*

$$\sum_{m \leqslant X} r_K(m) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ \mathrm{Nr}(\mathfrak{a}) \leqslant X}} 1 = \frac{2^{r_1}(2\pi)^{r_2} h(\mathcal{O}_K)\mathrm{reg}(\mathcal{O}_K)}{w_K |\mathrm{disc}(\mathcal{O}_K)|^{1/2}} X + O_K(X^{1-1/n}).$$

In this formula $h(\mathcal{O}_K) = |\mathrm{Cl}(\mathcal{O}_K)| \in \mathbb{N}_{>0}$ is the class number, $\mathrm{reg}(\mathcal{O}_K) > 0$ is the regulator, $\mathrm{disc}(\mathcal{O}_K) \in \mathbb{Z}$ is the discriminant and $w_K = |\mu_K|$ is the size of the group of roots of unity in K.

Reduction:

$$\sum_{m \leq X} r_K(m) = \sum_{Nr(\mathfrak{a}) \leq X} 1 = \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} \sum_{\substack{\mathfrak{b} \in [\mathfrak{a}] \\ Nr(\mathfrak{b}) \leq X}} 1$$

$$= \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} \sum_{\substack{\mathfrak{b} \in [\mathfrak{a}]^{-1} \\ Nr(\mathfrak{b}) \leq X}} 1$$

$$\mathfrak{b} \in [\mathfrak{a}]^{-1} \iff a\mathcal{O}_K^\times \quad a \in \mathfrak{a} - \{0\}$$

$$a\mathcal{O}_K = \mathfrak{a}.\mathfrak{b} \qquad a' \in a\mathcal{O}_K^\times \quad a'\mathcal{O}_K = a\mathcal{O}_K$$

$$|Nr(a)| = Nr(\mathfrak{a})\,Nr(\mathfrak{b})$$

$$\sum_{\substack{\mathfrak{b} \in [\mathfrak{a}]^{-1} \\ Nr(\mathfrak{b}) \leq X}} 1 = \sum_{\substack{a\mathcal{O}_K^\times \subset \mathfrak{a} - \{0\} \\ |Nr(a)| \leq Nr(\mathfrak{a})X}} 1$$

$$K_\infty^\times = \mathbb{R}^{\times r_1} \times \mathbb{C}^{\times r_2} \ni z = (x_1, \dots, x_{r_1}, z_1, \dots z_{r_2})$$

$$K_{\infty, \leq X}^\times = \left\{ z \in K_\infty^\times \quad Nr(z) = \prod |x_i| \prod |z_j|^2 \leq X \right\}$$

$\mathcal{F}$ = fund domain for $K_\infty^\times / \sigma_\infty(\mathcal{O}_K^\times)$

$\mathcal{F}_{\leq X}$ = ——————— $K_{\infty, \leq X}^\times / \sigma_\infty(\mathcal{O}_K^\times)$

Count

$$\Longrightarrow \sum_{\substack{a \in \mathfrak{A} - \{0\} \\ \sigma_\infty(a) \in \mathcal{J}_{\leq X}}} 1$$

$$\sigma_\infty(a) \in \mathcal{J}_{\leq X}$$

$\bigcap$

$\sigma_\infty(\mathfrak{A})$ = a lattice in K_∞

Lipshitz Principle

$\Omega \subset \mathbb{R}^n$ precompact, $t > 0$

$\Lambda \subset \mathbb{R}^n$ a lattice

$$\left| \Lambda \cap t\,\Omega \right| = \frac{\mathrm{vol}(t\Omega)}{\mathrm{vol}(\Lambda)} + \underset{t \to \infty}{\text{Err Term}}$$

$$= \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\Lambda)} t^n + \text{Err Term}$$

$\partial\Omega$ needs to be "regular" enough

Definition 4.9. *Let $\varphi : X \to Y$ be a map between metric spaces. The map φ is Lipschitz if there exists $c \geqslant 0$ such that*

$$\forall x, x' \in X,\ d(\varphi(x), \varphi(x')) \leqslant c.d(x, x').$$

Definition 4.10. *A compact domain $\Omega \subset \mathbb{R}^n$ has Lipschitz boundary if its boundary $\partial\Omega$ is the union of the images of a finite set of Lipschitz maps*

$$\varphi : [0,1]^{n-1} \to \partial\Omega.$$

Theorem 4.11 (Lipschitz principle). *Let Λ be a lattice and Ω compact with Lipschitz boundary. We have as $t \to \infty$*

$$N_\Omega(t, \Lambda) = \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\Lambda)} t^n + O_{\Lambda,\Omega}(t^{n-1}).$$

$$|\Lambda \cap t\Omega|$$

Proof: We can do an linear transformation mapping the canonical basis of $\mathbb{Z}^n$ to a basis of Λ. It preserve "beings Lipshitz"

wlog we that $\Lambda = \mathbb{Z}^n$

We count $|\mathbb{Z}^n \cap t\Omega|$

$$\mathbb{R}^n = \bigsqcup_{\lambda \in \mathbb{Z}^n} C(\lambda) \qquad C(\lambda) = \prod_{i=1}^{n} [\lambda_i, \lambda_i + 1[$$

$$\Omega_t = t\Omega = \bigsqcup_{\lambda \in \mathbb{Z}^n} \Omega_t \cap C(\lambda)$$

$$|\{\lambda \in \mathbb{Z}^n \ C(\lambda) \subset \Omega_t\}| \leq |\mathbb{Z}^n \cap \Omega_t| \leq |\{\lambda \ C(\lambda) \cap \Omega_t \neq \emptyset\}|$$

$$\mid \qquad\qquad \mid \leq \mathrm{vol}(\Omega_t) \leq \mid \qquad\qquad \mid$$

It is sufficient to show that

$$0 \leq |\{\lambda \text{ st } C(\lambda) \cap \Omega_t \neq \phi\}| - |\{\lambda \text{ st } C(\lambda) \subset \Omega_t\}| \overset{?}{\ll} t^{n-1}$$

Suppose $\lambda \in \{\lambda \text{ st } C(\lambda) \cap \Omega_t \neq \phi\} \setminus \{\lambda \; C(\lambda) \subset \Omega_t\}$

then λ is at distance $\leq \operatorname{diam} C(\lambda) = \sqrt{n} = O(1)$ of from a point of $\partial\Omega_t$

Supposons Ω is covered by

$$\varphi_1([0,1[^{n-1}), \dots \varphi_d([0,1[^{n-1})$$

for φ_i Lipshitz maps then

$$t\varphi_1([0,1[^{n-1}), \dots t.\varphi_d([0,1[^{n-1})$$ cover

$$\Omega_t = t\Omega$$

Let $P_t \in \partial \Omega_t$ be the points of the shape

$$t\varphi_1(\eta/t), \dots, t\varphi_d(\eta/t) \quad \eta \in \mathbb{Z}^{n-1} \cap [0,t[^{n}$$

$|P_t| = O(t^{n-1})$ and any point in $[0,1[^{n-1}$

is at distance $\leq 1/t$ from some η/t

Since the φ_i are Lipshitz any point of $\partial\Omega_t$ is at distance $O(1)$ from a point in P_t. $\Rightarrow$ the number of λ is $O(t^{n-1})$

$\square$

<u>Rmq</u>: the proof also give
$$|\Lambda \cap \partial\Omega_t| = O(t^{n-1}) \text{ at } t \to \infty$$

Corollary 4.2. *Notations and assumptions being as above, we have as* $t \to \infty$

$$N_{\partial\Omega}(t, \Lambda) = |\{\lambda \in \Lambda,\ \ \lambda \in \partial(\Omega_t)\}| = O(t^{n-1}).$$

Application to the CNF

$$\left| \left\{ a \in \mathfrak{a} - \{0\} \quad Nr(a) \leq X\, Nr(\mathfrak{a}) \right\} / \mathcal{O}_K^\times \right|$$

$$\mathcal{O}_K^\times = \mu_K \times U \qquad U \simeq \mathbb{Z}^{r-1}$$

group generated by fundamental units.

$$\left| \left\{ a \in \mathfrak{a} - \{0\} \quad Nr(a) \leqslant X\, Nr(\mathfrak{a}) \right\} / \mathcal{O}_K^\times \right|$$

$$= \frac{1}{w_K} \left| \left\{ a \in \mathfrak{a} - \{0\} \quad Nr(a) \leqslant X\, Nr(\mathfrak{a}) \right\} / U \right|$$

$w_K = |\mu_K|$

Let $\mathcal{F}_{\leq X}$ a fund domaine for

a fund domain for $K_\infty^\times{}_{\leq X} / U$

Recall that

$$\mathrm{Log}_\infty(U) = \mathrm{Log} \circ \sigma_\infty(U) \text{ is a lattice}$$

in $H(\mathbb{R}) = \left\{ (l_1, \dots, l_r) \in \mathbb{R}^r \; l_1 + \dots + l_r = 0 \right\}$

Let $\mathcal{P}_U = \sum_{i=1}^{r-1} [0,1[\log_\infty(\varepsilon_i)$

$$U = \varepsilon_1^{\mathbb{Z}} \times \varepsilon_2^{\mathbb{Z}} \times \dots \times \varepsilon_{r-1}^{\mathbb{Z}}$$

Now $\mathrm{Log}^{-1}(\mathcal{P}_U) = \mathcal{F}_1$ is a fundamental domain for $K^\times_{\infty,1}/U$

$$K^\times_{\infty,1} = \{ z \in K_\infty^\times \ \ \mathrm{Nr}(z) = 1 \}$$

$$\mathcal{F}_{\leq X} =]0, X^{1/n}[\cdot \mathcal{F}_1 = \left\{ tz \ \ \begin{array}{l} t \in]0, X^{1/n}[\\ z \in K^\times_{\infty,1} \\ \mathrm{Log}\, z \in \mathcal{P}_U \end{array} \right\}$$

is a fund domain for $K^{\times}_{\infty, \leq X} / U$

and $\overline{\mathcal{F}}_{\leq X}$ has Lipshitz boundary.

$$\left| \sigma_\infty(\mathfrak{a}) \cap \overline{\mathcal{F}}_{\leq X} \right| = \frac{\mathrm{vol}(\overline{\mathcal{F}}_{\leq 1})}{\mathrm{vol}(\sigma_\infty(\mathfrak{a}))} X + O_{\sigma_\infty(\mathfrak{a})}\left(X^{1-\frac{1}{n}}\right)$$

$$= \frac{\mathrm{vol}(\overline{\mathcal{F}}_{\leq 1}) 2^{r_2}}{|\mathrm{disc}\, \mathcal{O}_K|^{\frac{1}{2}} \mathrm{Nr}(\mathfrak{a})} X + O\left(X^{1-\frac{1}{n}}\right)$$

Add the contributions of the various representatives $\mathfrak{a}$ of $Cl(\mathcal{O}_K) = \{[\mathfrak{a}]\}$

$$\longrightarrow \left|\{\mathfrak{b} \subset \mathcal{O}_K \quad Nr(\mathfrak{b}) \leq X\}\right|$$

$$= \sum_{[\mathfrak{a}]} \sum_{\substack{a\mathcal{O}_K^\times \quad a \in \mathfrak{a}-\{0\} \\ Nr(a) \leq X\, Nr(\mathfrak{a})}} 1$$

$$= \sum_{[\mathfrak{a}]} \frac{\operatorname{vol}(\mathcal{F}_{\leq 1})\, 2^{r_2}}{|\operatorname{disc} \mathcal{O}_K|^{1/2}\, \operatorname{Nr}(\mathfrak{a})} X \operatorname{Nr}(\mathfrak{a}) + O\left(X^{1-\frac{1}{n}}\right)$$

$$= \frac{\operatorname{vol}(\mathcal{F}_{\leq 1})\, 2^{r_2}}{|\operatorname{disc} \mathcal{O}_K|^{1/2}} h(\mathcal{O}_K) X + O\left(X^{1-\frac{1}{n}}\right)$$

PROPOSITION 4.13. *We have*

$$\mathrm{vol}(\mathcal{F}_{\leqslant 1}) = 2^{r_1 - r_2}(2\pi)^{r_2}\mathrm{reg}(\mathcal{O}_K)$$

Proof: See Course Notes.

<u>Rmq</u> (Uniform distribution in ideal Classes)

The proof give

$$\left|\left\{ \mathfrak{b} \subset [\mathfrak{a}] \quad Nr(\mathfrak{b}) \leq X \right\}\right|$$

$$= \frac{2^{r_1}(2\pi)^{r_2} \operatorname{vol}(\mathcal{O}_K)}{w_K |\operatorname{disc}\mathcal{O}_K|^{\frac{1}{2}}} X + O_{\mathfrak{a}}\left(X^{1-\frac{1}{n}}\right)$$

<u>Simple analog</u> : $q \geq 1 \quad a \in \mathbb{Z}/q\mathbb{Z}$

$$\left|\left\{ n \leq X \quad n \equiv a\ (q) \right\}\right| = \frac{X}{q} + O(1)$$

as $X \to \infty$.

$n = qk + a$

count the k's

Dedekind ζ function

$$r_K(m) = \left|\{ \mathfrak{a} \subset \mathcal{O}_K \quad Nr(\mathfrak{a}) = m \}\right|$$

$$\forall \varepsilon > 0 \quad r_K(m) = O_\varepsilon(m^\varepsilon)$$

$$\sum_{m \geq 1} \frac{r_K(m)}{m^s}$$ is abs converging for $\operatorname{Re} s > 1$

$$= \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{Nr(\mathfrak{a})^s} = \zeta_K(s)$$

PROPOSITION 4.14. *Let s be a complex number with $\Re s > 1$ the series*

$$\zeta_K(s) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{\mathrm{Nr}(\mathfrak{a})^s}$$

converges absolutely and defines an holomorphic function in the half-plane $\Re s > 1$.

$\zeta_K(s)$ = Dedekind Zeta function of K

Rmq: $K = \mathbb{Q}$ $\quad \zeta_{\mathbb{Q}}(s) = \sum_{m \geq 1} \frac{1}{m^s} = \zeta(s) =$ Riemann ζ.

PROPOSITION 4.15 (Euler product formula). *For $\Re s > 1$ we have the identity of holomorphic functions*

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - \frac{1}{\mathrm{Nr}(\mathfrak{p})^s})^{-1}$$

Since $r_K(m)$ is multiplicative for $\mathrm{Re}\, s>1$

$$\sum_{m\geq 1} \frac{r_K(m)}{m^s} = \prod_p \left(\sum_{\alpha\geq 0} \frac{r_K(p^\alpha)}{p^{\alpha s}} \right)$$

$$\sum_{\alpha \geq 0} \frac{r_K(p^\alpha)}{p^{\alpha s}} = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ Nr(\mathfrak{a}) = \text{power of } p}} \frac{1}{Nr(\mathfrak{a})^s}$$

The ideals of norm p^α for some α are divisible by prime ideal above p

$$\sum_{\mathfrak{p} | p} \sum_{\alpha_\mathfrak{p} \geq 0} \frac{1}{Nr\left(\prod_{\mathfrak{p}|p} \mathfrak{p}^{\alpha_\mathfrak{p}}\right)^s} = \sum_{\mathfrak{p}|p \ \alpha_\mathfrak{p} \geq 0} \frac{1}{\prod_{\mathfrak{p}|p} Nr(\mathfrak{p})^{\alpha_\mathfrak{p} s}}$$

$$\prod_{\mathfrak{p}|p}\left(\sum_{\alpha_{\mathfrak{p}}\geq 0}\frac{1}{Nr(\mathfrak{p})^{\alpha_{\mathfrak{p}}s}}\right)=\prod_{\mathfrak{p}|p}\left(1-\frac{1}{Nr(\mathfrak{p})^{s}}\right)^{-1}$$

$$\zeta_K(s)=\prod_{p}\prod_{\mathfrak{p}|p}\left(1-\frac{1}{Nr(\mathfrak{p})^{s}}\right)^{-1}$$

$$=\prod_{\mathfrak{p}}\left(1-\frac{1}{Nr(\mathfrak{p})^{s}}\right)^{-1}$$

THEOREM 4.12. *The Dedekind ζ function admits meromorphic continuation to the half-plane $\{s,\ \Re s > 1 - 1/n\}$ with a simple pole at $s = 1$. We have*

$$\operatorname{res}_{s=1} \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h(\mathcal{O}_K)\operatorname{reg}(\mathcal{O}_K)}{w_K |\operatorname{disc}(\mathcal{O}_K)|^{1/2}}$$

where $w_K = |\mu_K|$. In particular $\zeta_K(s) \neq 0$ for $\Re s > 1$.

Proof: Set $\rho = \dfrac{2^{r_1}(2\pi)^{r_2} h(\mathcal{O}_K)\operatorname{reg}(\mathcal{O}_K)}{w_K |\operatorname{disc}\mathcal{O}_K|^{1/2}}$

$$r_{K,0}(m) = r_K(m) - \rho$$

By the Class # formula we have

$$\sum_{m \le X} r_{0,k}(m) = \sum_{m \le X} r_k(m) - \rho \sum_{m \le X} 1$$

$$= \rho X + O\left(X^{1-\frac{1}{n}}\right) - \rho X + O(1)$$

$$= O\left(X^{1-\frac{1}{n}}\right)$$

Let $\sigma > 1 - \frac{1}{n}$

$$\sum_{m\le X}\frac{r_{K,0}(m)}{m^{\sigma}} = \left[x^{-\sigma}\sum_{m\le x} r_{K,0}(m)\right]_{\frac{1}{2}}^{X} + \sigma\int_{\frac{1}{2}}^{X} x^{-\sigma-1}\left(\sum_{m\le x} r_{K,0}(m)\right)dx$$

$$= O\left(1 + X^{1-\frac{1}{n}-\sigma}\right) + O\left(\int_{1/2}^{X} x^{-\frac{1}{n}-\sigma}\,dx\right)$$

If $\sigma > 1 - \frac{1}{n}$ this bounded as $X \to \infty$

$\Rightarrow \sum \frac{r_{K,0}(m)}{m^s}$ absolutely converging for $\mathrm{Re}\, s > 1 - \frac{1}{n}$ and holomorphic there.

$$\sum_{m \leq X} \frac{r_{K,0}(m)}{m^s} = \zeta_{K,X}(s) - \rho\, \zeta_X(s)$$

$$\zeta_{K,X}(s) = \sum_{m \le X} \frac{r_K(m)}{m^s} \qquad \zeta_X(s) = \sum_{m \le X} \frac{1}{m^s}$$

if $\mathrm{Re}\, s > 1$ as $X \to \infty$ the partial sums converge

to $\zeta_K(s)$ and $\zeta(s)$

$\zeta_K(s)$ has analytic cont to $\mathrm{Re}\, s > 1 - \frac{1}{n}$

with a pole at $s=1$ with residu

$$\rho \cdot \mathrm{Res}_{s=1} \zeta(s) = \rho$$

$$\zeta_K(s) - \rho\,\zeta(s) = \sum_{m\geq 1} \frac{r_{K,0}(m)}{m^s}$$

is analytic for $\operatorname{Re} s > 0$ with a pole at $s=1$ residue ρ.

is analytic $\operatorname{Re} s > 1 - \frac{1}{n}$

<u>Example</u>: $\zeta_q = \exp\left(\frac{2\pi i}{q}\right)$ $K_q = \mathbb{Q}(\zeta_q)$

K_q = q-th cyclotomic field

Galois ext of $\mathbb{Q}$ of degree $\varphi(q) = \left|\left(\mathbb{Z}/q\mathbb{Z}\right)^\times\right|$

$\mathrm{Gal}(K_q/\mathbb{Q}) \simeq \left(\mathbb{Z}/q\mathbb{Z}\right)^\times$

We know that $\mathcal{O}_{K_q} = \mathbb{Z}[\zeta_q]$

$$\zeta_{K_q}(s) = \zeta(s) \prod_{\substack{\chi(q) \\ \chi \neq 1}} L(\chi, s)$$

χ varies over the character mod q

$$\chi : \left(\mathbb{Z}/q\mathbb{Z}\right)^{\times} \to \mathbb{C}^{\times}$$

$$L(\chi, s) = \sum_{m \geq 1} \frac{\chi(m)}{m^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

Distribution of prime ideals

We wait an asymptotic for:

$$\left|\left\{ \mathfrak{p} \subset \mathcal{O}_K \text{ prime } Nr(\mathfrak{p}) \leq X \right\}\right|$$

$K = \mathbb{Q}$ $\quad \left|\left\{ p \leq X \ \ p \text{ prime} \right\}\right| \sim \dfrac{X}{\log X}$

Prime Number Theorem

involve the analytic proprerties

of $\zeta(s)$

THEOREM 4.14 (Mertens Theorem). *As $X \to \infty$ we have*

$$\sum_{p \leqslant X} \frac{\log p}{p} = \log(X) + O(1).$$

and much more involved in the

THEOREM 4.15 (Prime number theorem).

$$\pi(X) = |\{p \textit{ prime}\ ,\ p \leqslant X\}| \simeq X/\log X$$

(1) Analytic cont of $\zeta(s)$ for $\operatorname{Re} s > 1-\varepsilon$ $\varepsilon > 0$

(2) $\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$

(3) $\zeta(s) \neq 0$ when $\operatorname{Re} s = 1$

THEOREM 4.16 (Mertens Theorem over K). *As $X \to \infty$ we have*

$$\sum_{\mathrm{Nr}(\mathfrak{p}) \leqslant X} \frac{\log \mathrm{Nr}(\mathfrak{p})}{\mathrm{Nr}(\mathfrak{p})} = \log(X) + O(1).$$

and much harder is the *Prime Number Theorem for number fields.*

THEOREM 4.17 (Prime number theorem over K). *As $X \to \infty$ we have*

$$\sum_{\mathrm{Nr}(\mathfrak{p}) \leqslant X} \log \mathrm{Nr}(\mathfrak{p}) = X + o(X).$$

and

$$\sum_{\mathrm{Nr}(\mathfrak{p}) \leqslant X} 1 = \frac{X}{\log X} + o(\frac{X}{\log X}).$$

(1) Analytic Cont of $\zeta_K(s)$ to $\mathrm{Re}\, s > 1 - \frac{1}{n}$

(2) $\prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathrm{Nr}(\mathfrak{p})^s}\right)^{-1}$

(3) $\zeta_K(s) \neq 0$ for $\mathrm{Re}\, s = 1$

Primes in Ideal Classes

We have seen that each ideal contains asymptotically the same # of ideals of norm $\leq X$ as $X \to \infty$

Is it true for the prime ideals?

Yes!

THEOREM 4.17 (à la Dirichlet). *For any class* $[\mathfrak{a}]$, *one has*

$$\lim_{s\to 1^+}\sum_{\mathfrak{p}\in[\mathfrak{a}]}\frac{\log \mathrm{Nr}(\mathfrak{p})}{\mathrm{Nr}(\mathfrak{p})^s}\Big/\lim_{s\to 1^+}\sum_{\mathfrak{p}}\frac{\log \mathrm{Nr}(\mathfrak{p})}{\mathrm{Nr}(\mathfrak{p})^s}=1/h(\mathcal{O}_K).$$

In particular there are infinitely many prime ideals in a given ideal class.

THEOREM 4.18 (à la Mertens). *For any class* $[\mathfrak{a}]$, *one has*

$$\sum_{\substack{\mathrm{Nr}(\mathfrak{p})\leqslant X\\ \mathfrak{p}\in[\mathfrak{a}]}}\frac{\log \mathrm{Nr}(\mathfrak{p})}{\mathrm{Nr}(\mathfrak{p})}=\frac{1}{h(\mathcal{O}_K)}\log X+O(1)$$

THEOREM 4.19 (à la Hadamard de la Vallee-Poussin). *For any class* $[\mathfrak{a}]$, *one has*

$$\sum_{\substack{\mathrm{Nr}(\mathfrak{p})\leqslant X\\ \mathfrak{p}\in[\mathfrak{a}]}}\log \mathrm{Nr}(\mathfrak{p})=\frac{1}{h(\mathcal{O}_K)}X+o(X)$$

$$\sum_{\substack{\mathrm{Nr}(\mathfrak{p})\leq X\\ \mathfrak{p}\in[\mathfrak{a}]}}1=\frac{1}{h(\mathcal{O}_K)}\frac{X}{\log X}+o\left(\frac{X}{\log X}\right)$$

Analogous to Dirichlet Thm on primes in arithmetic progressions

$q \geq 1 \quad (a,q)=1$

$$\pi(X,q,a) = \left| \{ p \leq X \quad p \equiv a \, (q) \} \right|$$

$$= \frac{1}{\varphi(q)} \frac{X}{\log X} + o\left(\frac{X}{\log X} \right)$$

Class Group L-fcts

Dirichlet L-fcts: $q \geq 1$

$$\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$$

$$L(\chi, s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

Class Grp L-fcts

$$\chi : Cl(\mathcal{O}_K) \to \mathbb{C}^\times$$

$$L(\chi, s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\chi([\mathfrak{a}])}{Nr(\mathfrak{a})^s} \qquad \mathrm{Re}\, s > 1$$

$$= \prod_{\mathfrak{p}} \left(1 - \frac{\chi([\mathfrak{p}])}{Nr(\mathfrak{p})^s}\right)^{-1}$$

THEOREM 4.13. *If χ is not trivial $L(\chi, s)$ admits analytic continuation to $\Re s > 1 - 1/n$ (with no poles in this domain).*

Proof: $\chi \neq 1$

$$\sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ Nr(\mathfrak{a}) \leq X}} \chi([\mathfrak{a}]) = \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} \chi([\mathfrak{a}]) \sum_{\substack{\mathfrak{a}' \subset \mathcal{O}_K \\ Nr(\mathfrak{a}') \leq X \\ \mathfrak{a}' \in [\mathfrak{a}]}} 1$$

$$= \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} \chi([\mathfrak{a}]) \left(\frac{\rho}{h(\mathcal{O}_K)} X + O\left(X^{1-\frac{1}{n}}\right) \right)$$

$$= \frac{\rho}{h(O_K)} \cdot X \underbrace{\sum_{[\mathfrak{a}] \in Cl(O_K)} \chi([\mathfrak{a}])}_{\substack{\shortparallel \\ 0}} + O\left(X^{1-\frac{1}{n}}\right)$$

Since $\chi \neq 1$

$$\sum_{\substack{\mathfrak{a} \subset O_K \\ Nr(\mathfrak{a}) \leq X}} \chi([\mathfrak{a}]) = O\left(X^{1-\frac{1}{n}}\right)$$

$$\sum_{\substack{\mathfrak{a} \in G_K \\ Nr(\mathfrak{a}) \leq X}} \frac{\chi([\mathfrak{a}])}{Nr(\mathfrak{a})^s} = \text{Abel Sommation by Part}$$

as $X \to \infty$

the partial sum is converging for

$$\operatorname{Re} s > 1 - \frac{1}{n}$$

$\Longrightarrow$ $L(\chi, s)$ is holomorphic for $\mathrm{Re}\, s > 1 - \frac{1}{n}$.

To get Mertens type thm one need to show that $L(\chi, 1) \neq 0$.