Week 5

Recall: We introduced c-convexity, c-subdifferential, c-transform.

- $\varphi(x) + \varphi^c(y) + c(x,y) \geq 0 \qquad \forall x \in X, y \in Y.$
- $\varphi(x) + \varphi^c(y) + c(x,y) = 0 \iff y \in \partial^c \varphi(x).$

Kantorovich duality $c : X \times Y \to \mathbb{R}$ continuous, bounded below.

If $\displaystyle\inf_{\gamma \in \Gamma(\mu,\nu)} \int c \, d\gamma = (KP) < +\infty$, then

$$(KP) = \max_{\substack{\varphi,\psi :\\ \varphi(x)+\psi(y)+c(x,y)\geq 0}} \int_X -\varphi \, d\mu + \int_Y -\psi \, d\nu = (DP)$$

Remark: to show $(\leq)$ we actually prove that $\operatorname{supp} \bar\gamma$ c-cycl. monotone $\Rightarrow$

$$\exists \varphi \text{ c-convex} : \int c(x,y) \, d\bar\gamma(x,y) = \int -\varphi \, d\mu + \int -\varphi^c \, d\nu \leq (DP) = (KP).$$

We conclude that: $\bar\gamma$ has c-cyclic. monotone supp $\iff$ $\bar\gamma$ is optimal in (KP)

Corollary: Let c be continuous, bounded below. Then, it is equivalent:

- $\bar\gamma$ is optimal in (KP).
- $\operatorname{supp} \bar\gamma$ is c-cycl. monotone.
- $\exists \varphi$ c-convex s.t. $\operatorname{supp} \bar\gamma \subseteq \partial^c \varphi$

Corollary: if c l.s.c. bounded below, then $(KP) = (DP)$ still holds

(but maybe the max. in (DP) is a sup).

Proof: $(\geq)$ it is easy, from the same proof.

$(\leq)$ wlog $c \geq 0$. Take $c_n \geq 0$, $c_n \uparrow c$, c_n continuous.

From theorem,

$$\underbrace{\min_{\gamma \in \Gamma(\mu,\nu)} \int c_n \, d\gamma}_{\int c_n \, d\overline{\gamma_n} \;\overset{!!}{=}} = \sup_{\varphi(x)+\psi(y) \geq -c_n(x,y)} \int -\varphi \, d\mu + \int -\psi \, d\nu$$

$$\leq \sup_{\varphi(x)+\psi(y) \geq -c(x,y)} \int -\varphi \, d\mu + \int -\psi \, d\nu$$

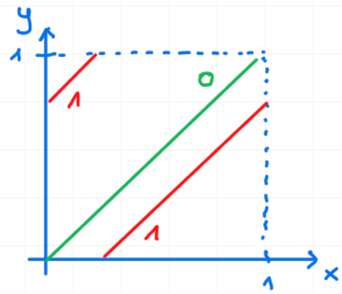
$\bar{\gamma}_n \in \Gamma(\mu,\nu)$, up to subsequences, $\bar{\gamma}_n \longrightarrow \bar{\gamma}$ narrowly.

So, $\forall n_0$

$$\int c_{n_0}\, d\bar{\gamma} = \lim_{n\to\infty} \int c_{n_0}\, d\bar{\gamma}_n \leq \underbrace{\liminf_{n\to\infty} \int c_n\, d\bar{\gamma}_n}_{\text{independent of } n_0} .$$

Take sup in n_0 and use monotone convergence theorem :

$$(KP) \leq \int c\, d\bar{\gamma} \leq \liminf_{n\to\infty} \int c_n\, d\bar{\gamma}_n \leq (DP).$$



<u>Example</u>: Let $\alpha \in [0,1] \setminus \mathbb{Q}$.
$$\underbrace{c(x,y)}_{\text{is l.s.c. but not continuous}} = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } y = x-\alpha \\ +\infty & \text{otherwise} \end{cases}$$

$\mu = \nu = \mathcal{L}^1_{[0,1]}$

Then: (A) $T_0(x) = x$ is optimal in (KP)?

(?) (B) $T_1(x) = \begin{cases} x-\alpha & \text{if } 1 > x > \alpha \\ x+1-\alpha & \text{if } 0 < x < \alpha \end{cases}$ is optimal in (KP)?

(C) $\mathrm{supp}\,(Id, T_0)_\# \mathcal{L}_{[0,1]}$ is c-cycl. monotone?

(D) $\mathrm{supp}\,(Id, T_1)_\# \mathcal{L}_{[0,1]}$ is c-cycl. monotone?

Real life problems involving optimal transport (Santambrogio's Section 1.7.3) (Villani, Old & New, Section 5).

① (KP): X = skills of the employees

μ = (number of) distribution of employees with skill x

Y = tasks to distribute

ν = distribution of tasks

$p(x,y)$ productivity of employee with skill x when performing task y.

Goal of the company : $\max\left\{ \int p(x,y)\, d\gamma : \gamma \in \Gamma(\mu,\nu)\right\}$.

② Kantorovich duality : (X,μ) distribution of bakeries

(Y,ν) distribution of coffee shops

Bakery x sells bread to coffee shop y at cost $c(x,y)$.

A consortium wants to organize the market.

It buys for price $\overbrace{\psi_0(x)}^{\text{price per unit}}$ all bread of x, sells to y at price $\psi_1(y)$:

Maximum profit: $\displaystyle\max_{-\psi_0(x)+\psi_1(y)\le c(x,y)} \left\{ -\int \psi_0(x)\, d\mu(x) + \int \psi_1(y)\, d\nu(y) \right\} =$

$$\underset{\text{theorem}}{=} \min\left\{ \int c(x,y)\, d\gamma(x,y) : \gamma \in \Gamma(\mu,\nu)\right\}.$$

Remark: Another (formal) proof of $(KP) = (DP)$:

Lagrange multiplier: it is 0 if $\pi_1\#\gamma = \mu$, $+\infty$ otherwise.

$$\inf_{\gamma\in\Gamma(\mu,\nu)} \int_{X\times Y} c(x,y)\, d\gamma = \inf_{\gamma\ge 0}\left\{ \int c\, d\gamma + \sup_{\varphi}\left(\int_{X\times Y} \varphi(x)\, d\gamma(x,y) - \int_X \varphi(x)\, d\mu(x)\right) + \sup_{\psi}\left(\int_{X\times Y} \psi(y)\, d\gamma(x,y) - \int_Y \psi(y)\, d\nu(y)\right)\right\}$$

$$= \inf_{\gamma}\ \sup_{\varphi,\psi} \left\{ \int c\, d\gamma + \int \varphi(x)\, d\gamma - \int \varphi(x)\, d\mu + \int \psi(y)\, d\gamma - \int \psi(y)\, d\nu \right\} = (*)$$

we can exchange:

<u>Lemma (Sion's minimax theorem)</u>: X compact, convex topological space, Y convex subset of linear space, $f : X\times Y \to \mathbb{R}$ continuous, <u>convex in x</u>, <u>concave in y</u>; then:

$$\min_{x\in X}\ \sup_{y\in Y} f(x,y) = \sup_{y\in Y}\ \min_{x\in X} f(x,y).$$

take $\gamma = M\,\delta_{\bar x,\bar y}$, $\underset{+\infty}{\underbrace{M}}$

$$\begin{cases} \equiv 0 & \text{if } c(x,y)+\varphi(x)+\psi(y)\geq 0 \ \forall (x,y)\in X\times Y \\ \equiv -\infty & \text{if } \exists \bar x,\bar y : c(\bar x,\bar y)+\varphi(\bar x)+\psi(\bar y)<0 . \end{cases}$$

$$(*) = \sup_{\varphi,\psi} \left\{ -\int \varphi\, d\mu - \int \psi\, d\nu + \inf_{\gamma\geq 0} \int_{X\times Y} [c(x,y)+\varphi(x)+\psi(y)]\, d\gamma \right\}$$

$$= \sup_{\varphi(x)+\psi(y)+c(x,y)\geq 0} \int -\varphi\, d\mu + \int -\psi\, d\nu .$$

<u>Exercise</u>: Let $c(x,y) = \frac{1}{2}|x-y|^2$, φ convex, μ-a.e. differentiable.

Let $\mu \in \mathcal{P}(\mathbb{R}^d)$. Is $\nabla\varphi$ optimal between μ and $(\nabla\varphi)_{\#}\mu$?

Graph $\nabla\varphi$ is closed and $\supseteq$ supp $(Id, \nabla\varphi)_{\#}\mu$ $\forall \mu$.

$\hookrightarrow$ c-cycl. monotone $\equiv \partial\varphi$.

By a previous Corollary, plan supported on a c-cycl. monotone set is optimal.

<u>Brenier's theorem</u>: Let $X = Y = \mathbb{R}^d$, $c(x,y) = \frac{1}{2}|x-y|^2$, let $\mu,\nu \in \mathcal{P}(\mathbb{R}^d)$,

with $\int |x|^2 d\mu + \int |y|^2 d\nu < +\infty$, $\mu \ll \mathcal{L}^d$.

Then, there exists a <u>unique</u> optimizer $\bar\gamma$ in (KP).

In addition, $\exists\, T = \nabla\varphi$, φ convex, s.t. $\bar\gamma = (Id, T)_{\#}\mu$,

$T = \nabla\varphi$ is the unique optimizer in (MP).

<u>Idea of the proof</u>:

(∃) Let $\bar\gamma$ be an optimizer in (KP). By Corollary, supp $\bar\gamma$ is c-cycl. monotone,

supp $\bar\gamma \subseteq \partial\varphi$, φ convex. If $\varphi \in C^1 \Rightarrow$ supp $\bar\gamma \subseteq$ graph $\nabla\varphi$.

We claim $\bar\gamma = (Id, \nabla\varphi)_{\#}\mu$: $\forall F$ measurable ≥ 0,

$$\int F(x,y)\, d\bar\gamma(x,y) = \int F(x,\nabla\varphi(x))\, d\bar\gamma(x,y) = \int F(x,\nabla\varphi(x))\, d\mu(x) =$$

$$= \int F(x,y)\, d(Id,\nabla\varphi)_{\#}\mu .$$

$\hookrightarrow$ $F(x,y) = F(x,\nabla\varphi(x))$ a.e. on supp $\bar\gamma \subseteq$ graph $\nabla\varphi$

(!) Let $\gamma_1 \neq \gamma_2$ optimal in (KP).

$\Rightarrow \frac{\gamma_1+\gamma_2}{2}$ also optimal $\Rightarrow \gamma_1, \gamma_2, \frac{\gamma_1+\gamma_2}{2}$ are supported on the graph of a convex function.

This is a contradiction, because the union of two (different) graphs is not a graph.

<u>Example</u>: What is the OT map (wrt quadratic cost) between the constant measures μ_1 and μ_2 supported on B_1 and $B_2 \setminus B_1$, in $\mathbb{R}^2$?