Week 4

Recall:

$\boxed{\Rightarrow}$ we only saw this

Theorem: γ optimal in (KP) $\Leftrightarrow$ $\operatorname{supp}\gamma$ is $\underbrace{\text{c-cycl. monotone}}_{\sum_{i=1}^{N} c(x_i,y_i) \le \sum_{i=1}^{N} c(x_{i+1},y_i) \ \forall (x_i,y_i)_{i=1,\dots,N} \in \operatorname{supp}\gamma}$

Rmk: $c(x,y) = \begin{cases} \frac{|x-y|^2}{2} \\ -x\cdot y \end{cases}$ have same minimizers in (KP)

Theorem (Rockafellar): $\left(c(x,y) = \frac{|x-y|^2}{2}\right)$

$S \subseteq \mathbb{R}^d \times \mathbb{R}^d$ is c-cyclically monotone $\Leftrightarrow$ $\exists\, \varphi$ convex $: \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, $\varphi \not\equiv +\infty$, such that $S \subseteq \partial\varphi$

$\partial\varphi = \bigcup_x \{x\} \times \partial\varphi(x)$, $\partial\varphi(x) := \{y \in \mathbb{R}^d : \forall z \in \mathbb{R}^d,\ \varphi(z) \geq \varphi(x) + \langle y, z-x\rangle\}$.

Proof (Rockafellar):

easy one $\leftarrow$ $\boxed{\Leftarrow}$ Assume $S \subseteq \partial\varphi$, take $\{(x_i,y_i)\}_{i=1,\dots,N} \subset S \subseteq \partial\varphi$, i.e.,

$$\varphi(x) \geq \varphi(x_i) + \langle y_i, x - x_i\rangle \quad \forall x.$$

I want: $\sum_{i=1}^{N} \langle y_i, x_{i+1} - x_i\rangle \leq 0$. Take $x = x_{i+1}$, $x_{N+1} = x_1$, $\sum_i$.

$$\sum_i \varphi(x_{i+1}) \geq \sum_i \varphi(x_i) + \langle y_i, x_{i+1} - x_i\rangle. \quad ✓$$

$\boxed{\Rightarrow}$ Let $(x_0,y_0) \in S$ fixed. Our φ will need to satisfy, for every $\{(x_i,y_i)\}_{i=1,\dots,N} \subseteq S$,

$$\begin{aligned}
\varphi(x) &\overset{(x_N,y_N)\in\partial\varphi}{\geq} \langle y_N, x - x_N\rangle + \varphi(x_N) \\
&\geq \langle y_N, x - x_N\rangle + \langle y_{N-1}, x_N - x_{N-1}\rangle + \varphi(x_{N-1}) \\
&\geq \dots \geq \langle y_N, x-x_N\rangle + \langle y_{N-1}, x_N - x_{N-1}\rangle + \dots + \langle y_0, x_1 - x_0\rangle + \varphi(x_0)
\end{aligned}$$

$\partial\varphi$ is invariant under $\varphi \mapsto \varphi + c$, $c \in \mathbb{R}$

A good definition of φ is:

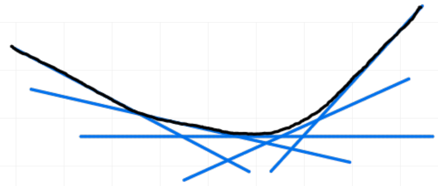
$$\varphi(x) = \sup\{\langle y_N, x - x_N\rangle + \dots + \langle y_0, x_1 - x_0\rangle : N \geq 1,\ (x_i,y_i) \in S\ \forall i = 1,\dots,N\}$$

fixed

important 1 to N !

(x_0,y_0) fixed

- φ is convex (sup of affine functions)



- Take $(\bar{x}, \bar{y}) \in S$. We claim:

$$\varphi(z) \geq \langle \bar{y}, z - \bar{x} \rangle + \varphi(\bar{x}) \quad \forall z \in \mathbb{R}^d \quad (*)$$

Indeed, let $\alpha < \varphi(\bar{x})$, and by definition, $\exists N \geq 1$, $(x_1, y_1), \dots, (x_N, y_N) \in S$ such that $\langle y_N, \bar{x} - x_N \rangle + \dots + \langle y_0, x_1 - y_0 \rangle \geq \alpha$.

Consider $(x_i, y_i)_{i=1,\dots,N+1}$ with $x_{N+1} = \bar{x}$, $y_{N+1} = \bar{y}$, admissible in the definition of φ:

$$\varphi(z) \geq \langle \underset{\bar{y}}{\underbrace{y_{N+1}}}, z - x_{N+1} \rangle + \langle y_N, \underset{\bar{x}}{\underbrace{x_{N+1}}} - x_N \rangle + \dots + \langle y_0, x_1 - x_0 \rangle$$

$$\geq \langle \bar{y}, z - \bar{x} \rangle + \alpha$$

Since α is arbitrarily close to $\varphi(\bar{x})$, we conclude $(*)$.

⚠ we have not yet used c-cyclical monotonicity! We don't know $\varphi \not\equiv +\infty$.

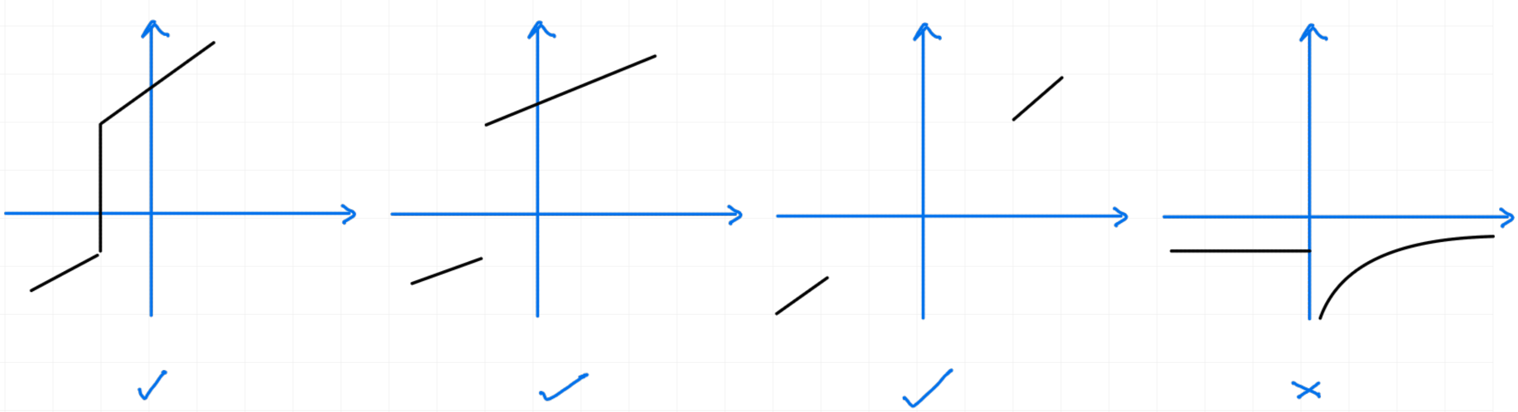
- $\varphi(x_0) \underset{N=1}{\geq} \langle y_1, x_0 - x_1 \rangle + \langle y_0, x_1 - x_0 \rangle = 0 \quad ((x_1, y_1) = (x_0, y_0))$

- $\varphi(x_0) \leq 0$, because $\forall (x_i, y_i) \in S$, $N \geq 1$,

$$\langle y_N, x_0 - x_N \rangle + \dots + \langle y_0, x_1 - x_0 \rangle \leq 0.$$

Hence, $\varphi \not\equiv +\infty$. $\square$

<u>Exercise</u>: What is c-cyclically monotone in $\mathbb{R}^2$?



A tool from convex analysis :

this definition is typically given for convex functions, even if it makes sense for every function

Def: Let $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, $(\varphi \not\equiv +\infty)$, the Legendre transform of φ is $\varphi^* : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$:

$$\varphi^*(y) := \sup_{x \in \mathbb{R}^d} \{ x \cdot y - \varphi(x) \}$$

Convex functions are the supremum of affine functions, $\varphi(x) := \sup_{y \in \mathbb{R}^d} \{ \underbrace{x \cdot y}_{-c(x,y)} + \xi(y) \}$

↳ for every slope y we choose a "height" $\xi(y)$.

Def: Given X, Y metric spaces, $c : X \times Y \to \mathbb{R}$, $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, φ is c-convex if :

$$\varphi(x) = \sup_{y \in \mathbb{R}^d} \{ -c(x,y) + \xi(y) \}$$

for some ξ.

⚠ IMORTANT : Santambrogio uses a different sign

Def: Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$. The c-(Legendre) transform of φ, denoted φ^c, is

$$\varphi^c : Y \to \mathbb{R} \cup \{+\infty\}$$
$$y \mapsto \varphi^c(y) := \sup_{x \in \mathbb{R}^d} \{ -c(x,y) - \varphi(x) \}$$

- φ is c-convex $\iff$ $\varphi = \xi^c$ for some ξ.

Exercise (in analogy with Series) : $(\varphi^c)^c \leq \varphi$. Equality holds $\iff$ φ is c-convex .

Def: Given $\varphi : X \to \mathbb{R}$ c-convex, the c-subdifferential of φ at $x \in X$ is :

$$\partial^c \varphi(x) := \{ y \in Y \;:\; \forall z \in X,\ \varphi(z) \geq \varphi(x) + \underbrace{c(x,y) - c(z,y)}_{\langle y, z-x \rangle \text{ before}} \}$$

Let $$\partial^c \varphi := \bigcup_x \{x\} \times \partial^c \varphi(x)$$

Theorem : $S \subseteq X \times Y$ is c-cyclically monotone $\iff \exists\, \varphi$ c-convex, $\varphi \not\equiv +\infty$ s.t. $S \subseteq \partial^c \varphi$

Proof : Literally copy-paste the previous proof :

$\boxed{\Leftarrow}$ Assume $S \subseteq \partial^c \varphi$, take $\{(x_i, y_i)\}_{i=1,\dots,N} \subseteq S \subseteq \partial^c \varphi$,

$\varphi(x) \geq \varphi(x_i) + c(x_i, y_i) - c(z, y_i)$ $\forall x$. I want :

$\sum_{i=1}^{N} [c(x_{i+1}, y_i) - c(x_i, y_i)] \geq 0$ take $x = x_{i+1}$, $x_{N+1} = x_1$, same as before.

$\boxed{\Rightarrow}$ Define:

$$\varphi(x) := \sup \left\{ \begin{array}{c} -c(x, y_N) + c(x_N, y_N) - c(x_{N-1}, y_N) + \dots + c(x_0, y_0) : \\ N \geq 1, \ (x_i, y_i) \in S \ \forall i = 1, \dots, N \end{array} \right\}$$

Same properties as before. $\square$

Remark: If $X = Y$, $c(x,y) = d(x,y)$, φ is c-convex $\Leftrightarrow$ $\underbrace{\varphi \text{ is } 1\text{-Lipschitz}}_{|\varphi(x) - \varphi(y)| \leq |x-y|}$.

Indeed, if φ 1-Lipschitz:

$$\varphi(x) \geq \varphi(y) - d(x,y) \ . \Rightarrow \varphi(x) \geq \sup_y \{\varphi(y) - d(x,y)\} \overset{x=y}{\geq} \varphi(x)$$

$$\Rightarrow \varphi(x) = \sup_y \{\varphi(y) - d(x,y)\} \Rightarrow \varphi \text{ is c-convex.}$$

Conversely, if φ is c-convex $\left(\varphi(x) = \sup_z \{\xi(z) - d(x,z)\} \right)$ fix $x, y \in X$:

$$\underset{\forall \varepsilon > 0, \exists z: \varphi(y) \leq \xi(z) - d(z,y) + \varepsilon}{-\varphi(x) + \varphi(y) \leq} \underbrace{-d(z,y) + \xi(z) + \varepsilon}\ \underbrace{- \xi(z) + d(x,z)}_{\varphi(x) \geq \xi(z) - d(x,z)} \overset{\text{triangle inequality}}{\leq} d(x,y) + \varepsilon$$

ε is arbitrary $\Rightarrow -\varphi(x) + \varphi(y) \leq d(x,y)$. Changing roles $\Rightarrow \varphi$ 1-Lip. $\square$

Prop: Given φ c-convex:

- $\varphi(x) + \varphi^c(y) + c(x,y) \geq 0 \qquad \forall x \in X, y \in Y.$
- $\varphi(x) + \varphi^c(y) + c(x,y) = 0 \qquad \Leftrightarrow y \in \partial^c \varphi(x).$

Proof:
- $\varphi^c(y) \geq -c(x,y) - \varphi(x)$.
- $y \in \partial^c \varphi(x) \Leftrightarrow \varphi(z) \geq \varphi(x) + c(x,y) - c(z,y) \ \forall z \Leftrightarrow$

$$\Leftrightarrow -\varphi(x) - c(x,y) \geq \sup_z \{-c(z,y) - \varphi(z)\} = \varphi^c(y)$$

$\leq$ works by first point. $\square$

Theorem (Kantorovich duality): Assume $c : X \times Y \to \mathbb{R}$ continuous, bounded below (equivalently, ≥ 0). If $\inf_{\gamma \in \Gamma(\mu,\nu)} \int c \, d\gamma = (KP) < +\infty$, then

$$(KP) = \max_{\varphi, \psi \,:\, \varphi(x)+\psi(y)+c(x,y) \geq 0} \int_X -\varphi \, d\mu + \int_Y -\psi \, d\nu =: (DP)$$

Remark: $c(x,y) = |x-y|^2$ in $\mathbb{R}^d$, $(KP) < +\infty$ as soon as:

$\int |x|^2 d\mu + \int |y|^2 d\nu < +\infty$. Indeed, $\gamma = \mu \otimes \nu \in \Gamma(\mu,\nu)$ has finite cost.

Proof:

$\boxed{(KP) \geq (DP)}$: Take φ, ψ admissible for (DP),

$$\int c(x,y) \, d\gamma(x,y) \geq \int -\varphi(x) \, d\gamma + \int -\psi(y) \, d\gamma = \int -\varphi \, d\mu + \int -\psi \, d\nu .$$

Take sup over admissible φ, ψ, and inf over $\gamma \in \Gamma(\mu,\nu) \Rightarrow (KP) \geq (DP)$.

$\boxed{(KP) \leq (DP)}$ Let $\bar\gamma \in \Gamma(\mu,\nu)$ optimal $\Rightarrow \operatorname{supp} \bar\gamma$ is c-cycl. monotone $\Rightarrow$ by the previous theorem, $\exists\, \varphi$ c-convex s.t. $\operatorname{supp}\bar\gamma \subset \partial^c \varphi$.

By Proposition above $\Rightarrow$ φ, φ^c admissible in the dual problem, and $-\varphi(x) - \varphi^c(y) = c(x,y) \quad \forall (x,y) \in \operatorname{supp}\bar\gamma$:

$$\int c(x,y) \, d\bar\gamma = \int -\varphi(x) \, d\bar\gamma + \int -\varphi^c(y) \, d\bar\gamma = \int -\varphi \, d\mu + \int -\varphi^c \, d\nu$$

Hence, $(\bar\gamma, \varphi, \varphi^c)$ gives equality (KP) and (DP). $\square$

Remark: the optimality of $\bar\gamma$ is used only to deduce $\operatorname{supp}\bar\gamma \subseteq \partial^c\varphi$.

Remark: $X = Y$, $c(x,y) = d(x,y)$, φ c-convex $\Leftrightarrow$ φ 1-Lipschitz, $\varphi^c = -\varphi$:

$$\min_{\gamma \in \Gamma(\mu,\nu)} \int_{X \times X} d(x,y) \, d\gamma = \max_{\varphi \text{ 1-Lipschitz}} \int_X \varphi \, (d\mu - d\nu).$$