Week 13

Recall:
The heat equation is the GF of $\int |\nabla u|^2$ in L^2 and of the entropy, $\int \rho \log \rho$, in Wasserstein.

Define given $\tau > 0$, $\rho_k^\tau dx$ is the min in $\mathcal{P}(\Omega)$ of $\rho \mapsto \frac{W_2^2(\rho, \rho_{k-1}^\tau)}{2\tau} + \int \rho \log \rho$

[we saw a lemma on existence, absol. continuity, and does not concentrate on $\partial\Omega$]

$\rho^\tau(t,x) = \rho_k^\tau$

Theorem (Jordan, Kinderlehrer, Otto; 1998) (JKO scheme)

There exists $\rho \in L^1_{loc}([0,+\infty)\times\Omega)$ such that $\rho^\tau \longrightarrow \rho$ up to a subsequence and ρ is a solution to the heat equation starting from ρ_0.

Lemma: $\forall \xi \in C^\infty(\Omega; \mathbb{R}^d)$ tangent to $\partial\Omega$,

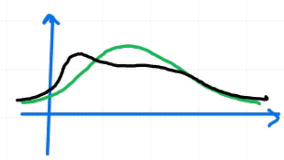
T_{k+1} is the OT map from ρ_k^τ to ρ_{k+1}^τ.

$$\int_\Omega \rho_{k+1}^\tau \operatorname{div} \xi \, dx = \frac{1}{\tau} \int_\Omega \langle \xi \circ T_{k+1}, T_{k+1} - x \rangle \rho_k^\tau \, dx$$

Proof:
$$\frac{W_2^2(\rho_{k+1}^\tau, \rho_k^\tau)}{2\tau} + \int \rho_{k+1}^\tau \log \rho_{k+1}^\tau \le \underbrace{\frac{W_2^2(\rho_\varepsilon, \rho_k^\tau)}{2\tau}}_{(I)} + \underbrace{\int \rho_\varepsilon \log \rho_\varepsilon}_{(II)} \qquad (*)$$

How to define a competitor ρ_ε?

① Outer variation: $\rho_{k+1}^\tau + \varepsilon \rho_0(x)$, with $\int \rho_0 = 0$



② Inner variation: Let ξ as in the statement, $\phi_\varepsilon(x) = x + \varepsilon \xi(x)$,

$\rho_\varepsilon = \phi_\varepsilon \# \rho_{k+1}^\tau$.

③ Inner variation (better done): Let ϕ_ε be the flow at time ε of ξ:

$$\begin{cases} \partial_t \phi(t,x) = \xi(\phi(t,x)) \\ \phi(0,x) = x \end{cases}$$

(it is tangent to the boundary).

$g(T(x)) \det \nabla T(x) = f(x)$

$$\rho_\varepsilon = \phi_\varepsilon \# \rho_{k+1}^\tau \iff \rho_{k+1}^\tau(x) = \rho_\varepsilon(\phi_\varepsilon(x)) \det \nabla \phi_\varepsilon(x)$$

$$\phi_\varepsilon(x) = x + \varepsilon \xi(x) + o(\varepsilon)$$

$$\nabla \phi_\varepsilon(x) = Id + \varepsilon \nabla \xi(x) + o(\varepsilon) \Longrightarrow \det \nabla \phi_\varepsilon(x) = 1 + \varepsilon \operatorname{div} \xi + o(\varepsilon)$$

$$\hookrightarrow \begin{pmatrix} 1 + \varepsilon \partial_1 \xi & \varepsilon \partial_2 \xi \\ \varepsilon \partial_1 \xi & 1 + \varepsilon \partial_2 \xi \end{pmatrix}$$

$$\int \rho_\varepsilon(y)\log\rho_\varepsilon(y)\,dy = \int \rho_\varepsilon(\phi_\varepsilon(x))\log\rho_\varepsilon(\phi_\varepsilon(x))\det\nabla\phi_\varepsilon(x)\,dx$$

$$= \int \rho_{k+1}^{2}\left(\log\rho_{k+1}^{2} - \log\det\nabla\phi_\varepsilon(x)\right)dx$$

$$= \int \rho_{k+1}^{2}\log\rho_{k+1}^{2} - \varepsilon\int\rho_{k+1}^{2}\operatorname{div}\xi\,dx + o(\varepsilon).$$

Let T_{k+1} be the OT map. A transport map between ρ_k^2 and ρ_ε is $\phi_\varepsilon\circ T_{k+1}$.

$$\frac{W_2^2(\rho_\varepsilon,\rho_k^2)}{2\tau} \le \frac{1}{2\tau}\int |\underbrace{\phi_\varepsilon\circ T_{k+1}(x)}_{T_{k+1}(x)+\varepsilon\xi(T_{k+1}(x))} - x|^2\rho_k^2(x)\,dx$$

$$\le \frac{1}{2\tau}\int\left[|T_{k+1}(x)-x|^2 + 2\varepsilon\langle T_{k+1}(x)-x,\xi(T_{k+1}(x))\rangle\right]\rho_k^2(x)\,dx + o(\varepsilon)$$

$$= \frac{1}{2\tau}W_2^2(\rho_{k+1}^2,\rho_k^2) + \frac{2\varepsilon}{2\tau}\int\langle T_{k+1}(x)-x,\xi(T_{k+1}(x))\rangle\rho_k^2(x)\,dx + o(\varepsilon)$$

Plugging these in $(*)$:

$$0 \le -\varepsilon\int\rho_{k+1}^2\operatorname{div}\xi\,dx + \frac{\varepsilon}{\tau}\int\langle T_{k+1}(x)-x,\xi(T_{k+1}(x))\rangle\rho_k^2(x)\,dx + o(\varepsilon).$$

Divide by ε, let $\varepsilon\downarrow 0$, and change ξ by $\overset{\text{still tangent to } \partial\Omega}{-\xi}$ to get the desired result.

Def: $\rho\in L^1_{loc}$ satisfies the heat equation (HE) distributionally starting from ρ_0 if $\forall\,\psi\in C_c^\infty(\Omega)$, $\xi\in C_c^\infty([0,\infty))$ we have:

$$\xi(0)\int_\Omega\psi\rho_0\,dx + \int_0^\infty\int_\Omega\rho_t\left(\psi(x)\partial_t\xi(t) + \Delta\psi(x)\xi(t)\right)dx\,dt = 0$$

- with Neumann boundary conditions if $\psi\in C^\infty(\Omega)$ with $\frac{\partial\psi}{\partial\nu}=0$ on $\partial\Omega$ instead.

$\to$ definition to avoid regularity issues ($\rho\in L^1_{loc}$).

Remark: it is actually equivalent to being a classical solution.

Remark 2: Why is it a reasonable definition?

If $\rho\in C^2$ solves $\begin{cases}\partial_t\rho - \Delta\rho = 0 & \text{in } \Omega\times[0,\infty)\\ \frac{\partial\rho}{\partial\nu}=0 & \text{on } \partial\Omega\end{cases}$ then it is satisfied.

Indeed, test the PDE with $\psi(x)\xi(t)$: $\int_0^\infty\int_\Omega\big(\underbrace{\partial_t\rho\,\psi(x)\xi(t)}_{(A)} - \underbrace{\Delta\rho\,\psi(x)\xi(t)}_{(B)}\big) = 0$

$$(A) = -\int_\Omega\psi(x)\int_0^\infty\rho\,\partial_t\xi(t) - \int_\Omega\psi(x)\xi(0)\rho(0,x)$$

(B) We know: $\nabla\rho\cdot\nabla\psi + \Delta\rho\,\psi = \operatorname{div}(\nabla\rho\,\psi) \longrightarrow$ integrated is $\int_{\partial\Omega}\overbrace{\partial_\nu\rho}^{0}\,\psi = 0$

thus $(B) = \int_0^\infty\int_\Omega\nabla\rho\cdot\nabla\psi\,\xi(t) = -\int_0^\infty\int_\Omega\Delta\psi\,\rho$

$$\int_{\partial\Omega}\overset{0}{\partial_\nu\psi}\,\rho = 0 \longleftarrow \operatorname{div}(\nabla\psi\,\rho) = \nabla\psi\cdot\nabla\rho + \Delta\psi\,\rho.$$

Proof of the main theorem (JKO):

ρ^τ_{k-1} is a competitor

$$\frac{W_2^2(\rho_k^\tau, \rho_{k-1}^\tau)}{2\tau} + \int \rho_k^\tau \log \rho_k^\tau \le \int \rho_{k-1}^\tau \log \rho_{k-1}^\tau \xrightarrow{\text{sum in } k}$$

$$\xrightarrow{\sum_k} \underbrace{\sum_{k=1}^{k_0} \frac{W_2^2(\rho_k^\tau, \rho_{k-1}^\tau)}{2\tau}}_{\geq 0} + \int \rho_{k_0}^\tau \log \rho_{k_0}^\tau \le \int \rho_0 \log \rho_0 \qquad (+)$$

measures cannot concentrate on $\partial\Omega$ or escape uniformly in t

$\Longrightarrow$ entropy decreasing in k: $\int \rho^\tau(t,x) \log \rho^\tau(t,x) \le \int \rho_0 \log \rho_0 \quad \forall t.$

Since $\int \rho^\tau(t,x)\,dx = 1 \longrightarrow \int_0^T \int_\Omega \rho^\tau(t,x) = T \ (**)$; and up to a subsequence, $\rho^\tau \longrightarrow \rho \in L^1$ in $L^1([0,T]\times\Omega)$ $\forall T>0$,

$\hookrightarrow$ why? same as last week, based on $(+)$.

and by $(**)$ in any time interval, $\int \rho = 1$ for a.e. $t \in [0,T]$.

By today's lemma, applied to $\xi = \nabla\varphi$, $\varphi \in C^\infty(\Omega)$, $\partial_\nu \varphi = 0$ on $\partial\Omega$,

$$-\int \Delta\varphi\, \rho_k^\tau\,dx + \frac{1}{\tau}\int_\Omega \underbrace{\langle \nabla\varphi \circ T_k, T_k - x\rangle}\, \rho_{k-1}^\tau\,dx = 0$$

$\hookrightarrow$ Taylor expansion: $\left| -\varphi(T_k) + \varphi(x) - \langle \nabla\varphi(T_k), x - T_k\rangle \right| \le \|D^2\varphi\|_{L^\infty} |x-T_k|^2$.

So: $$\left| -\int_\Omega \Delta\varphi\, \rho_k^\tau\,dx - \frac{1}{\tau}\int_\Omega \varphi \rho_{k-1}^\tau + \frac{1}{\tau}\int_\Omega \varphi \rho_k^\tau \right| \le \|D^2\varphi\|_{L^\infty} \frac{W_2^2(\rho_{k-1}^\tau, \rho_k^\tau)}{\tau}.$$

Now take $\zeta \in C_c^\infty([0,\infty))$ and multiply against $\tau\zeta((k-1)\tau)$,

$$\left| -\tau \int_\Omega \Delta\varphi\, \overbrace{\rho^\tau(k\tau,x)}^{\rho_k^\tau}\, \zeta((k-1)\tau) - \int_\Omega \varphi(x) \rho^\tau((k-1)\tau, x)\zeta((k-1)\tau) + \int_\Omega \varphi(x)\rho^\tau(k\tau,x)\zeta((k-1)\tau) \right|$$

$$\le \underbrace{\|D^2\varphi\|_{L^\infty} \|\zeta\|_{L^\infty}}_{\le C} W_2^2(\rho^\tau(k\tau,\cdot), \rho^\tau((k-1)\tau,\cdot)).$$

Sum over k:

$$\Bigg| \overbrace{-\tau \sum_{k=1}^\infty \int_\Omega \Delta\varphi\, \rho^\tau(k\tau,x)\zeta((k-1)\tau)}^{\mathrm{II}} - \zeta(0)\int_\Omega \varphi(x)\rho_0(x)\,dx \qquad (X)$$

$$\overbrace{+ \sum_{k=1}^\infty \int_\Omega \varphi(x) \rho^\tau(k\tau,x)\zeta((k-1)\tau) - \sum_{k=1}^\infty \int_\Omega \varphi(x)\rho^\tau(k\tau,x)\zeta(k\tau)}^{\mathrm{I}} \Bigg| \le$$

$$\le C \sum_{k=1}^\infty W_2^2(\rho^\tau(k\tau,\cdot), \rho^\tau((k-1)\tau,\cdot)) \le C\tau \longrightarrow 0 \text{ as } \tau \to 0$$

$\hookrightarrow (+)$

For I :

$$= \sum_{k=1}^{\infty} \int_{\Omega} \psi(x) \rho^{\tau}(\underset{t \in ((k-1)\tau, k\tau)}{t}, x) \Big[\overbrace{\xi((k-1)\tau) - \xi(k\tau)}^{\int_{(k-1)\tau}^{k\tau} \partial_t \xi} \Big] = - \int_0^{\infty} \int_{\Omega} \psi(x) \rho^{\tau}(t,x) \partial_t \xi(t) \, dx \, dt$$

as $\tau \to 0$, since $\rho^{\tau} \longrightarrow \rho$ weakly in L^1 this term goes to

$$\to \int_0^{\infty} \int_{\Omega} \psi(x) \rho(t,x) \partial_t \xi(t) \, dx \, dt .$$

For II : $\tau \xi((k-1)\tau) = \displaystyle\int_{(k-1)\tau}^{k\tau} \xi((k-1)\tau) \, dt \overset{\|\nabla \xi\|_{L^\infty} \leq C}{=} \int_{(k-1)\tau}^{k\tau} \xi(t) dt + O(\tau)$

$$\text{II} = - \int_0^{\infty} \int_{\Omega} \xi(t) \Delta \psi \rho^{\tau}(t,x) \, dx \, dt + O(\tau)$$

as $\tau \to 0$, this term goes to $\longrightarrow - \int_0^{\infty} \int_{\Omega} \xi(t) \Delta \psi \rho(t,x) \, dx \, dt$

Sending $\tau \downarrow 0$ in (X) we get :

$$\xi(0) \int_{\Omega} \psi \rho_0 \, dx + \int_0^{\infty} \int_{\Omega} \big[\xi(t) \Delta \psi(x) \rho(t,x) + \partial_t \xi(t) \psi(x) \rho(t,x) \big] = 0 .$$

This is the definition of distributional solution to the heat equation. $\square$