Week 12

Recall:

The gradient flow of ϕ, $x:[0,+\infty) \to H$, starting at x_0 is:

$$(GF) \begin{cases} \dot{x}(t) = -\nabla\phi(x(t)) \\ x(0) = x_0 \end{cases}$$

$\to$ if $\phi \notin \mathcal{C}^1$ but convex, we can replace by $\dot{x} \in -\partial\phi(x(t))$.

The implicit Euler scheme gives a way to approximate this problem. Given a fixed time-step τ, we define a "discrete solution" $x^\tau(t) = x_k^\tau$ if $t \in [(k-1)\tau, k\tau)$,

where $\begin{cases} x_0^\tau = x_0 \\ x_{k+1}^\tau \text{ solves } \frac{x - x_k^\tau}{2\tau} \in -\partial\phi(x). \end{cases}$ $\left(\text{or equivalently minimizes } \frac{\|x - x_k^\tau\|^2}{2\tau} + \phi(x)\right)$

$d(x, x_k^\tau)^2$ this makes sense in metric spaces

The Dirichlet energy functional

Let $H = L^2(\mathbb{R}^d)$, $\phi(u) = \begin{cases} \frac{1}{2}\int|\nabla u|^2 & \text{if } u \in W^{1,2}(\mathbb{R}^d) \\ +\infty & \text{otherwise.} \end{cases}$

Prop: The GF of ϕ wrt L^2 scalar product is the heat equation,

$$\partial_t \underbrace{u(t)}_{\in L^2(\mathbb{R}^d)} \in -\partial\phi(u(t)) \iff \partial_t u(t,x) = \Delta u(t,x) \text{ distributionally.}$$

(that is: $0 = \int(\partial_t\varphi(t,x)u(t,x) + \Delta\varphi(t,x)u(t,x))$ $\forall \varphi \in \mathcal{C}_c^\infty$.)

Proof: Claim: $\partial\phi(u) = \begin{cases} -\Delta u & \text{if } \Delta u \in L^2 \\ \emptyset & \text{otherwise.} \end{cases}$

$p \in \partial\phi(u) \iff \forall x = u + w,\ \phi(u+w) - \phi(u) \geq \langle p, w\rangle_{L^2}$,

$\forall u, v$: $\phi(u+w) - \phi(u) = \frac{1}{2}\int(|\nabla u + \nabla w|^2 - |\nabla u|^2) = \int\langle\nabla u, \nabla w\rangle + \frac{|\nabla w|^2}{2}$ $(*)$

- If $\Delta u \in L^2$, RHS of $(*)$ $= -\int \Delta u\, w + \overbrace{\int\frac{|\nabla w|^2}{2}}^{\geq 0} \geq \langle -\Delta u, w\rangle_{L^2} \Rightarrow -\Delta u \in \partial\phi(u)$.

- Let now $p \in \partial\phi(u)$. Then $\Delta u \in L^2$ and $p = -\Delta u$: indeed, $\forall w$,

$\int\langle\nabla u, \nabla w\rangle + \frac{|\nabla w|^2}{2} \geq \int p w$. Now fix w_0 and take $w = \varepsilon w_0$:

$$\cancel{\varepsilon}\int\langle\nabla u, \nabla w_0\rangle + \frac{\varepsilon^{\cancel{2}}}{2}\int|\nabla w_0|^2 \geq \cancel{\varepsilon}\int p w_0.$$ Exchanging w_0 with $-w_0$

we get equality, so $\forall w_0 \in \mathcal{C}_c^\infty$,

$$\underbrace{\int\langle\nabla u, \nabla w_0\rangle}_{\int(-\Delta u)w_0} = \int p w_0, \text{ then } -\Delta u = p \in L^2(\mathbb{R}^d). \qquad \square$$

as a distribution

Remark : Let ϕ be a convex, C^1 function. Then GF is unique :

Indeed, take $x(t), y(t)$ GF from x_0, y_0 :

$$\frac{1}{2}\frac{d}{dt}\|x(t)-y(t)\|^2 = \langle x(t)-y(t), \dot{x}(t)-\dot{y}(t)\rangle = (*) =$$

ϕ convex

$$= \langle x(t)-y(t), -\nabla\phi(x(t))+\nabla\phi(y(t))\rangle \leq 0$$

Applied with $x_0 = y_0$ we get uniqueness. subdifferential of convex is monotone.

If ϕ not C^1, $(*) = \langle x(t)-y(t), -p(t)+q(t)\rangle \leq 0$, $q(t) \in \partial\phi(y(t))$, $p(t) \in \partial\phi(x(t))$.

The heat equation as a gradient flow (#2)

A natural way to approximate the heat equation is

$$\underset{\displaystyle u_k^\tau}{\overset{\shortparallel}{u^\tau(\tau k)}} \text{ is the min in } L^2 \text{ of } \frac{\|u-u^\tau(\tau(k-1))\|_{L^2}^2}{2\tau} + \int|\nabla u|^2$$

u^τ is then extended piecewise constant.

.... The heat equation has another GF structure!

Define given $\tau>0$,

bounded convex domain in $\mathbb{R}^d$

$$\rho_k^\tau dx \text{ is the min in } \mathcal{P}(\Omega) \text{ of } \rho \mapsto \frac{W_2^2(\rho,\rho_{k-1}^\tau)}{2\tau} + \int\rho\log\rho$$

↳ slight abuse of notation, denoting ρ_k^τ both the measure and its density wrt Lebesgue.

Remark : • We can assume $\rho_0 \in \mathcal{P}(\Omega)$ by rescaling.

• We assume $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$ with $\int\rho_0\log\rho_0 < +\infty$

At τ fixed, we associate to the seq. $\rho^\tau(t,x) = \rho_k^\tau$ if $t \in ((k-1)\tau, k\tau)$

→ next week

Theorem (Jordan, Kinderlehrer, Otto ; 1998) :

There exists $\rho \in L^1_{loc}([0,+\infty)\times\Omega)$ such that

$\rho^\tau \to \rho$ up to a subsequence and ρ is a solution to the heat equation starting from ρ_0.

Lemma : Given $\bar{\rho} \in \mathcal{P}(\Omega)$ (open) s.t. $\int\bar{\rho}\log\bar{\rho} < +\infty$, then

$\exists\, \rho_\infty \in \mathcal{P}(\Omega)$, $\rho_\infty \ll dx$, min of $\rho \mapsto \dfrac{W_2^2(\rho,\bar{\rho})}{2\tau} + \int\rho\log\rho$

Proof: Let $\{\rho_m\}_m \subseteq \mathcal{P}(\Omega)$ be a minimizing sequence:

$$\frac{W_2^2(\rho_m, \bar\rho)}{2\tau} + \int \rho_m \log \rho_m \longrightarrow \inf_\rho \frac{W_2^2(\rho,\bar\rho)}{2\tau} + \int \rho \log \rho$$

absolut. continuous measure

$\{\rho_m\}_m \subseteq \mathcal{P}(\bar\Omega)$ $\underbrace{\text{are tight}}_{\text{defined on a compact set}}$ up to a subsequence, by Prokhorov: $\rho_m \xrightarrow{*} \rho_\infty$ measure, with $\int_{\bar\Omega} \rho_\infty = 1$

Claim 1: $\int_{\underbrace{\Omega}_{\neq \bar\Omega}} \rho_\infty = 1$ Claim 2: $\rho_\infty \ll dx$ (later)

This is nontrivial: a sequence bounded in $L^2(L^1) \Rightarrow$ any weak limit is L^2 (measure)

Conclusion of the proof: $s \mapsto s\log s$ convex implies
(with the two claims)

$$\int_\Omega \rho_\infty \log \rho_\infty \leq \liminf_m \int_\Omega \rho_m \log \rho_m \qquad \boxed{(1)}$$

$\hookrightarrow$ analogous to $\|\rho_\infty\|_{L^p} \leq \liminf_m \|\rho_m\|_{L^p}$, suggestion in lecture notes.

Look at $W_2^2(\rho_m, \bar\rho)$, $\rho_m \to \rho_\infty$:

$\rightarrow$ optimal between ρ_m and $\bar\rho \Rightarrow \gamma_m$ tight

stability theory

$$W_2^2(\rho_m,\bar\rho) = \int \underbrace{|x-y|^2}_{\text{continuous \& bdd in } \Omega\times\Omega} d\gamma_m \implies \gamma_{m_k} \longrightarrow \gamma \quad (\gamma \text{ transport plan between } \rho_\infty \text{ and } \bar\rho).$$

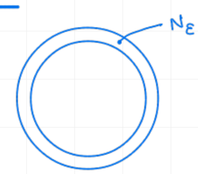
$$\Rightarrow \quad W_2^2(\rho_m,\bar\rho) \geq (W_2(\bar\rho,\rho_\infty) - \underbrace{W_2(\rho_m,\rho_\infty)}_{\to 0})^2 \qquad \boxed{(2)}$$

From $\boxed{(1)}$ and $\boxed{(2)}$:

$$\int \rho_\infty \log \rho_\infty + \frac{W_2^2(\rho_\infty,\bar\rho)}{2\tau} \leq \liminf_m \int \rho_m \log \rho_m + \frac{W_2^2(\rho_m,\bar\rho)}{2\tau}$$

by choice of ρ_m

Then ρ_∞ is a minimizer. $\qquad = \inf \int \rho \log \rho + \dfrac{W_2^2(\rho,\bar\rho)}{2\tau}$



Proof of claim 1: Let $N_\varepsilon = \{x : \text{dist}(x, \Omega^c) < \varepsilon\}$

For ε small, $|N_\varepsilon| < C\varepsilon$,

because $\int \rho_m \log \rho_m \leq \inf + 1$

$$\int_{N_\varepsilon} \rho_m \leq \int_{N_\varepsilon \cap \{\rho_m < L\}} \rho_m + \int_{N_\varepsilon \cap \{\rho_m \geq L\}} \rho_m \frac{\log \rho_m}{\log L} \leq LC\varepsilon + \frac{C}{\log L} \leq (*).$$

Choose L s.t. RHS small in ε, $L = \varepsilon^{-1/2}$,

$$(*) \leq \varepsilon^{1/2} + \frac{2C}{\log \varepsilon^{-1}} \leq \frac{C}{|\log \varepsilon|} \longrightarrow 0 \quad \text{as } \varepsilon \to 0 \text{ unif. in } m, \text{ so:}$$

cut-off $= \begin{cases} 1 & \Omega \setminus N_\varepsilon \\ 0 & N_{\varepsilon/2} \end{cases}$

$$\underset{\displaystyle\int \rho_\infty \varphi_\varepsilon}{\underset{\downarrow}{\int \rho_m \varphi_\varepsilon}} \geq \int_{\Omega\setminus N_\varepsilon} \rho_m \geq 1 - \frac{C}{|\log \varepsilon|}. \qquad \varepsilon \text{ arbitrary} \implies \int_\Omega \rho_\infty = 1.$$

Proof of claim 2: We know: $\int_{\Omega} \rho_\infty \log \rho_\infty < +\infty$.

$\rho_\infty \ll dx \iff \forall \varepsilon, \exists \delta$ such that $|A| < \delta \Rightarrow \int_A \rho_\infty dx < \varepsilon$.

But we have:

$$\int_A \rho_\infty dx \leq \int_{A \cap \{\rho_\infty < L\}} \rho_\infty dx + \int_{A \cap \{\rho_\infty \geq L\}} \rho_\infty \frac{\log \rho_\infty}{\log L} \leq L\delta + \frac{C}{\log L}.$$

As before, take $L = \delta^{-1/2}$ and we have:

$$\int_A \rho_\infty dx \leq \delta^{1/2} + \frac{C}{|\log \delta|} \leq \varepsilon \text{ if } \delta \text{ small enough.}$$