Week 10

Theorem: Let $X = \mathbb{R}^d$, $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$. Let γ optimal coupling for W_p. Then, a Wasserstein geodesic between μ and ν is $\boxed{\mu_t = \pi_t \# \gamma,}$ where $\pi_t(x,y) = (1-t)x + ty$.

Def: F is geodesically convex if $\forall x, y$, $\exists \eta : [0,1] \to X$ constant speed geodesic from x to y s.t. $F \circ \eta : [0,1] \to \mathbb{R} \cup \{+\infty\}$ is convex, i.e.,

$$F(\eta(t)) \le (1-t) F(x) + t F(y).$$

lower semi-continuous: $\mu_n \to \mu$ in $(\mathcal{P}_p(\mathbb{R}^d), W_p) \Rightarrow \mathcal{U}(\mu) \le \liminf_n \mathcal{U}(\mu_n)$

Convex and l.s.c. functionals on $\mathcal{P}_p(\mathbb{R}^d)$

Internal energy:

$$\mathcal{U}(\mu) = \begin{cases} \int u(\rho)\, dx & \text{if } \mu = \rho(x)\, dx \\ +\infty & \text{otherwise.} \end{cases}$$

Potential energy:

$$\mathcal{V}(\mu) = \int V \, d\mu, \quad V : \mathbb{R}^d \to \mathbb{R} \text{ given.}$$

Interaction energy:

$$\mathcal{W}(\mu) = \int W(x,y)\, d\mu(x)\, d\mu(y), \quad W : \mathbb{R}^d \to \mathbb{R} \text{ given.}$$

⚠ Convexity is NOT wrt the linear structure: $\left(\mathcal{U}(t\mu + (1-t)\nu) \cancel{\le} t\,\mathcal{U}(\mu) + (1-t)\,\mathcal{U}(\nu) \right)$ NO!

Displacement convexity

Def: Let X be a geodesic metric space, $F : X \to \mathbb{R} \cup \{+\infty\}$, $\lambda \in \mathbb{R}$.

F is λ-geodesically convex ($\underbrace{W_2\text{-convex}}_{\text{for measures}}$ or displacement convex; $\lambda = 0$) if $\forall x, y \in X$, $\exists \eta : [0,1] \to X$ constant speed geodesic between x and y such that: $F \circ \eta : [0,1] \to \mathbb{R}$ is λ-convex, i.e.,

$$F(\eta(t)) \le (1-t) F(x) + t F(y) - \frac{\lambda}{2} t(1-t) d^2(x,y) \quad \forall t \in [0,1].$$

Remark (λ-convexity in $\mathbb{R}^d$): $F : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ λ-convex $\iff$

$\iff F(x) - \frac{\lambda}{2}|x|^2$ is convex $\iff D^2 F \ge \lambda \, \mathrm{Id}$

Indeed : F λ-convex $\iff$ $F((1-t)x+ty) \leq (1-t)\left[F(x) - \frac{\lambda}{2}|x|^2\right]$

$$+\,t\left[F(y)-\frac{\lambda}{2}|y|^2\right] + \underbrace{\frac{t\lambda}{2}|y|^2 + (1-t)\frac{\lambda}{2}|x|^2 - \frac{\lambda}{2}t(1-t)|x-y|^2 - \frac{\lambda}{2}|(1-t)x+ty|^2}_{=\,0}$$

$$+\,\frac{\lambda}{2}|(1-t)x+ty|^2$$

$$\iff F - \frac{\lambda|x|^2}{2} \text{ convex } \iff D^2F - \lambda\,\mathrm{Id} \geq 0 .$$

Convexity of potential energy

Theorem : $V : \mathbb{R}^d \to \mathbb{R}^+ \cup \{+\infty\}$ is λ-convex $\iff$ $\mathcal{V} : \mathcal{P}_2(X) \to \mathbb{R}^+ \cup \{+\infty\}$ is λ-convex.

Proof : $\boxed{\Rightarrow}$ We claim that: $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, γ optimal between μ_0 and μ_1, μ_t geodesic $((1-t)\pi_1 + t\pi_2)_\# \gamma$.

Then $t \mapsto \int V\, d\mu_t$ is convex,

$$\int V\, d\mu_t = \int V((1-t)x+ty)\, d\gamma(x,y) \leq \int \left\{(1-t)V(x) + tV(y) - \frac{\lambda}{2}t(1-t)|x-y|^2\right\} d\gamma(x,y)$$

$$= (1-t)\int V(x)\, d\mu_0(x) + t\int V(y)\, d\mu_1(y) - \frac{\lambda}{2}t(1-t)W_2^2(\mu_0,\mu_1).$$

$\boxed{\Leftarrow}$ Apply convexity of $\mathcal{V}$ to δ_x and δ_y $\Rightarrow$ convexity of V.

Convexity of internal energy $\quad \int \mathcal{U}(\rho)\,dx \quad \rho \in \mathcal{P}_2$

(HP1) : $\begin{cases} \mathcal{U} : [0,\infty) \to \mathbb{R}\cup\{+\infty\} \text{ convex, l.s.c., } \mathcal{U}(0) = 0, \\ \limsup_{s \downarrow 0} \dfrac{\mathcal{U}^-(s)}{s^\alpha} < +\infty \text{ for some } \alpha > \dfrac{d}{d+2} \end{cases}$

negative part of $\mathcal{U}$, $\mathcal{U}^- = \max\{0, -\mathcal{U}\}$

d = dimension

(HP2) : $s \longrightarrow s^d\, \mathcal{U}(s^{-d})$ convex and decreasing on $(0,\infty)$.

we arbitrarily choose negative part finite, so that integral is well-defined.

Remark: if (HP1) holds, then $\int \mathcal{U}^-(\rho)\,dx < +\infty \quad \forall \rho \in L^1(\mathbb{R}^d)$.

Indeed, by (HP1) $\mathcal{U}^-(s) \leq C_1 s^\alpha + C_2 s$ far from 0, $\mathcal{U}$ convex

$$\int \mathcal{U}^-(\rho) \leq C_2 \overbrace{\int \rho}^{<\infty} + C_1 \int \left(\rho(1+|x|^2)\right)^\alpha \frac{1}{(1+|x|^2)^\alpha} \leq$$

$$\leq \bar{C} + C_1 \left\{ \left[\int \rho(1+|x|^2)\right]^\alpha \left[\int \frac{dx}{(1+|x|^2)^{\frac{\alpha}{1-\alpha}}}\right]^{1-\alpha} \right\} < \infty$$

integrable if $\frac{2\alpha}{1-\alpha} > d \iff \alpha > \frac{d}{d+2}$

NOTE: in this setting, the geodesic is unique; but this is not easy to see (see Figalli–Glaudo).

absolutely continuous measures

Theorem: If (HP1)–(HP2) hold, then $\mathcal{U}$ is convex along geodesics in $\mathcal{P}_{a,2}(\mathbb{R}^d)$

Remark: (HP1) & (HP2) hold for:

1. $\mathcal{U}(z) = z^\alpha, \ \alpha > 1.$

2. $\mathcal{U}(z) = z \log z$,

3. $\mathcal{U}(z) = -z^\alpha$

   $\hookrightarrow 1 \geq \alpha \geq 1 - \frac{1}{d}$

Proof: Let $T: \mu_0 \to \mu_1$ optimal map, T^{-1} inverse, assume T and T^{-1} smooth.

Let $\mu_t = T_t \# \mu_0$, $T_t = (1-t)I + tT$. Claim: $t \to \mathcal{U}(\mu_t)$ is convex for $t \in [0,1]$.

Step 1: $\mu_t = \rho_t \, dx$ with $\rho_t = \frac{\rho_0}{\underbrace{JT_t}_{= \det \nabla T_t}} \circ T_t^{-1}$.

Indeed, $\forall \varphi$:

$$\int \varphi \rho_t \, dx = \int \varphi \, d\mu_t = \int \varphi \circ T_t \, \rho_0 \overset{x = T_t^{-1}(y)}{=} \int \varphi \, \rho_0(T_t^{-1}(y)) \overbrace{|\det \nabla T_t^{-1}(y)|}^{\frac{1}{\det \nabla T_t} \circ T_t^{-1}} dy$$

Step 2: $t \mapsto \mathcal{U}(\mu_t) = \int \mathcal{U}\left(\frac{\rho_0}{JT_t} \circ T_t^{-1}\right) dx \overset{T_t^{-1}(x) = y}{=} \int \mathcal{U}\left(\frac{\rho_0}{JT_t}\right) JT_t \, dx$ is convex.

At x fixed, define: $t \xrightarrow[F]{} \underbrace{JT_t(x)^{1/d}}_{\geq 0} \xrightarrow[G]{} JT_t \, \mathcal{U}\left(\frac{\rho_0}{JT_t}\right)$

G is $G(s) = s^d \, \mathcal{U}\left(\frac{\rho_0(x)}{s^d}\right)$ which is convex and decreasing.

F concave $\iff F\left(\frac{t_1+t_2}{2}\right) \geq \frac{F(t_1)+F(t_2)}{2} \iff$

$$\iff \det{}^{1/d}\left(\Big(1 - \underbrace{\frac{t_1+t_2}{2}}_{\geq 0}\Big) I + \frac{t_1+t_2}{2} \underbrace{\nabla T}_{\geq 0}\right) \geq \det{}^{1/d}\left((1-t_1)I + t_1 \nabla T\right) + \det{}^{1/d}\left((1-t_2)I + t_2 \nabla T\right)$$

$\Leftarrow \det^{1/d}$ is a concave function (Step 3).

At fix x, $G \circ F$ is convex:

$$(G \circ F)\underbrace{((1-s)t_1 + s t_2)}_{\substack{\geq (1-s)F(t_1) + sF(t_2) \\ \downarrow \\ F \text{ concave}}} \overset{G \text{ decreasing}}{\leq} G((1-s)F(t_1) + sF(t_2)) \overset{G \text{ convex}}{\leq} (1-s)\, G \circ F(t_1) + s \, G \circ F(t_2).$$

Step 3: $A \longrightarrow (\det A)^{1/d}$ is concave, $A \in \mathrm{Sym}_{\geq 0}$ $d \times d$ matrix.

We will show: $\det((1-t)A_0 + tA_1)^{1/d} \geq (1-t)\det^{1/d} A_0 + t \det^{1/d} A_1$ (*).

We first reduce to having $A_0 = I$

A_0 symmetric $\longrightarrow A_0 = MDM^{-1}$, D diagonal, M orthonormal:

$\sqrt{A_0} := MD^{1/2}M^{-1}$, $D^{1/2} = \begin{pmatrix} d_1^{1/2} & & 0 \\ & \ddots & \\ 0 & & d_d^{1/2} \end{pmatrix}$ $\sqrt{A_0}\sqrt{A_0} = A_0$

$$(*) \Leftrightarrow \det^{1/d}\left(\cancel{A_0^{1/2}}\left[(1-t)I + t A_0^{-1/2} A_1 A_0^{-1/2}\right]\cancel{A_0^{1/2}}\right) \geq (1-t)\left(\cancel{\det^{1/d} A_0}\right) \oplus$$
$$+ \, t \det^{1/d}\left(\cancel{A_0^{1/2}} A_0^{-1/2} A_1 A_0^{-1/2} \cancel{A_0^{1/2}}\right)$$

and $A_0^{-1/2} A_1 A_0^{-1/2}$ is symmetric!

Diagonalize! Assume in $\oplus$ that $A_0^{-1/2} A_1 A_0^{-1/2}$ diagonal $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}$

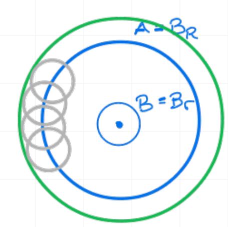
$$\oplus \Leftrightarrow \prod_{i=1}^{d} ((1-t) + t\lambda_i)^{1/d} \geq (1-t) + t \prod_{i=1}^{d} \lambda_i^{1/d} \Leftrightarrow$$

$$\Leftrightarrow 1 \geq (1-t) \cdot \underbrace{\left[\prod_{i=1}^{d} \frac{1}{1-t+t\lambda_i}\right]^{1/d}}_{\wedge\!\!|\ \ GM \leq AM} + t \underbrace{\left[\prod_{i=1}^{d} \frac{\lambda_i}{1-t+t\lambda_i}\right]^{1/d}}_{\wedge\!\!|}$$

$$\frac{1}{d}\sum_{i=1}^{d} \frac{1}{1-t+t\lambda_i} + \frac{1}{d}\sum_{i=1}^{d} \frac{\lambda_i}{1-t+t\lambda_i} = 1 \qquad \square$$

Application: Brunn-Minkowski inequality

Theorem: $A, B \subseteq \mathbb{R}^d$ compact. $\mathcal{L}^d(\underbrace{A+B}_{\text{still compact}})^{1/d} \geq \mathcal{L}^d(A)^{1/d} + \mathcal{L}^d(B)^{1/d}$ ⊗



equality case for Brunn-Minkowski!

Proof: Given $\mathcal{C} \subseteq \mathbb{R}^d$, we associate $\mu_C := \frac{1}{\mathcal{L}^d(C)} \mathbb{1}_C \, dx$.

Take $\mathcal{U}(\mu) = -\int \rho^{1-\frac{1}{d}} dx$ / ③ in previous remark.

$-\mathcal{U}(\mu_C) = \int_C \frac{dx}{\mathcal{L}^d(C)^{1-1/d}} = \mathcal{L}^d(C)^{1/d}$.

Let μ_t geodesic between μ_A and μ_B: $\overbrace{\frac{1}{2}[-\mathcal{U}(\mu_A) - \mathcal{U}(\mu_B)]}^{\text{RHS of ⊗}} \leq -\mathcal{U}(\mu_{1/2})$

NOTE: the geodesic is unique, but we do not prove it; see Figalli-Glaudo.

scale of space

We are now left to prove $-\mathcal{U}(\mu_{1/2}) \leq \mathcal{L}^d\left(\frac{A+B}{2}\right)^{1/d}$.

Claim: ρ probability supported on $\mathcal{C} \Rightarrow -\mathcal{U}(\rho) := \int \rho^{1-1/d} dx \leq \mathcal{L}^d(\mathcal{C})^{1/d}$.

With this claim we finish: $\mu_{1/2} = \frac{x + T(x)}{2} \# \mu_A$ is concentrated on $\frac{A+B}{2} = \mathcal{C}$.

on B

on A

concave $\varphi(\rho)$ probability Jensen

Proof of claim: $\int_{\mathcal{C}} \rho^{1-1/d} dx = \mathcal{L}^d(\mathcal{C}) \int_C \rho^{1-1/d} \frac{dx}{\mathcal{L}^d(\mathcal{C})} \leq \mathcal{L}^d(\mathcal{C}) \left[\int_{\mathcal{C}} \rho \frac{dx}{\mathcal{L}^d(\mathcal{C})}\right]^{1-1/d} = \mathcal{L}^d(\mathcal{C})^{1/d}$.