

Serie 3

Optimal transport, Fall semester

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Exercise 3.1. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a nonnegative lower semicontinuous function. Show that:

- (i) f admits a minimizer in every compact set $K \subset \mathbb{R}^n$.
- (ii) f can be approximated from below monotonically (namely, $f_\lambda(x) \uparrow f(x)$ for every $x \in \mathbb{R}^d$) by a sequence of functions f_λ as $\lambda \rightarrow \infty$, where f_λ is λ -Lipschitz.
- (iii) for every sequence of measures $\mu_n \rightharpoonup \mu$ narrowly,

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} f d\mu_n \geq \int_{\mathbb{R}^d} f d\mu.$$

Hint: For (ii), define $f_\lambda(x) = \inf_{y \in \mathbb{R}^n} \{f(y) + \lambda|x - y|\}$.

Solution:

- (i) Let $K \subset \mathbb{R}^d$ be a compact set and $\{x_n\}_{n=1}^\infty \subset K$ a minimizing sequence, i.e.

$$\lim_{n \rightarrow \infty} f(x_n) = \inf\{f(x) : x \in K\}.$$

Since K is compact, there is a converging subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that $x_{n_k} \rightarrow x \in K$. Then, since f is lower semicontinuous,

$$f(x) \leq \liminf_{k \rightarrow \infty} f(x_{n_k}) = \inf\{f(x) : x \in K\}.$$

Thus

$$f(x) \leq f(y) \quad \forall y \in K.$$

- (ii) Define $f_\lambda(x) = \inf_{y \in \mathbb{R}^n} \{f(y) + \lambda|x - y|\}$. We begin by showing that $\{f_\lambda\}_{\lambda \in \mathbb{R}}$ is increasing in λ and bounded by f . Assume $\lambda < \lambda'$, then

$$f_\lambda(x) \leq f(y) + \lambda|x - y| \leq f(y) + \lambda'|x - y| \quad \text{for every } y \in \mathbb{R}^d.$$

Taking the infimum over y , we get $f_\lambda(x) \leq f_{\lambda'}(x)$. In addition due to the definition of f_λ , we get $f_\lambda(x) \leq f(x)$. Now we prove that f_λ is indeed λ -Lipschitz. We have

$$f_\lambda(x') \leq f(y) + \lambda|x' - y| \leq f(y) + \lambda|x - y| + \lambda|x' - x|.$$

Taking the infimum over y , we get $f_\lambda(x') \leq f_\lambda(x) + \lambda|x' - x|$. In a similar way, we can prove $f_\lambda(x) \leq f_\lambda(x') + \lambda|x' - x|$. Hence, $|f_\lambda(x') - f_\lambda(x)| \leq \lambda|x - x'|$, for all $x, x' \in \mathbb{R}^d$, proving that f_λ is λ -Lipschitz.

Now finally, we prove that $\lim_{\lambda \rightarrow \infty} f_\lambda = f$. Let $x \in \mathbb{R}^d$. Since f is lower semicontinuous, for any $\epsilon > 0$, there is $\delta > 0$ so that for all y such that $|x - y| \leq \delta$ we have $f(y) \geq \min\{f(x) - \epsilon, \frac{1}{\epsilon}\}$. In addition, since f is nonnegative $f(y) + \lambda|x - y| \geq \delta\lambda$, for all y such that $|x - y| > \delta$. We conclude

$$f_\lambda(x) \geq \min\{f(x) - \epsilon, \frac{1}{\epsilon}, \delta\lambda\}.$$

Letting $\lambda \rightarrow \infty$, we get

$$\liminf_{\lambda \rightarrow \infty} f_\lambda(x) \geq \min\{f(x) - \epsilon, \frac{1}{\epsilon}\}.$$

Since ϵ is arbitrary, we deduce $\liminf_{\lambda \rightarrow \infty} f_\lambda(x) \geq f(x)$. Finally, since f_λ is bounded by f , we deduce

$$\lim_{\lambda \rightarrow \infty} f_\lambda(x) = f(x).$$

(iii) Notice that the function $\min\{f_\lambda, \lambda\}$ is continuous and bounded so that since $\mu_n \rightharpoonup \mu$,

$$\int_{\mathbb{R}^d} \min\{f_\lambda, \lambda\} d\mu = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \min\{f_\lambda, \lambda\} d\mu_n \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} f d\mu_n,$$

where the inequality follows from the fact that $f_\lambda \leq f$. Letting $\lambda \rightarrow \infty$, we get, due to the monotone convergence theorem,

$$\int_{\mathbb{R}^d} f d\mu \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} f d\mu_n.$$

Exercise 3.2. The support of a nonnegative measure $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ is defined as the smallest closed set on which μ is concentrated, i.e.

$$spt(\mu) := \bigcap \{C \subset \mathbb{R}^n \text{ closed} : \mu(\mathbb{R}^n \setminus C) = 0\}.$$

Let us take a sequence of nonnegative measures $\mu_j \in \mathcal{M}_+(\mathbb{R}^n)$ such that $\mu_j \xrightarrow{*} \mu$. Prove the following fact: for every $x \in spt(\mu)$, there exists a sequence of points $x_j \in spt(\mu_j)$ such that $x_j \rightarrow x$.

Solution: Suppose by contradiction that for some $\epsilon > 0$ there is a sequence $j_k \rightarrow \infty$ for which $spt(\mu_{j_k}) \subset \mathbb{R}^n \setminus B_\epsilon(x)$. Let us consider a test function $\varphi \in C_c(B_\epsilon(x))$ such that

$$\begin{cases} \varphi \geq 0 & \text{in } \mathbb{R}^n, \\ \varphi = 0 & \text{in } \mathbb{R}^n \setminus B_\epsilon(x), \\ \varphi = 1 & \text{in } B_{\epsilon/2}(x). \end{cases}$$

Since $x \in spt(\mu)$, $\mu(B_{\epsilon/2}(x)) > 0$. Testing the weak* convergence of μ_{j_k} to μ with φ , we get

$$0 = \int \varphi d\mu_{j_k} \xrightarrow{k \rightarrow \infty} \int \varphi d\mu \geq \mu(B_{\epsilon/2}(x)) > 0,$$

a contradiction.

Exercise 3.3 (Characterizations of weak-* convergence). Let $\mu_j, \mu \in \mathcal{P}(\mathbb{R}^n)$ be probability measures in $\mathbb{R}^n$.

i) Show that $\mu_j \rightarrow \mu$ narrowly if and only if one of the following properties hold:

a) For every open set $A \subset \mathbb{R}^n$:

$$\liminf_{j \rightarrow \infty} \mu_j(A) \geq \mu(A).$$

b) For every closed set $C \subset \mathbb{R}^n$:

$$\limsup_{j \rightarrow \infty} \mu_j(C) \leq \mu(C).$$

c) For every set $E \subset \mathbb{R}^n$ such that $\mu(\partial E) = 0$:

$$\lim_{j \rightarrow \infty} \mu_j(E) = \mu(E).$$

ii) Give an example of a sequence of probability measures $\mu_j \in \mathcal{P}(\mathbb{R}^n)$ such that $\mu_j \xrightarrow{*} \mu$ for some measure $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ and an open set A such that

$$\liminf_{j \rightarrow \infty} \mu_j(A) > \mu(A).$$

Hint: For one implication in (i), use Exercise 3.1. For the other, use the layer-cake formula

$$\int \varphi d\mu = \int_0^\infty \mu(\{\varphi > t\}) dt \quad \text{for every } \varphi \in C_b(\mathbb{R}^n), \varphi \geq 0.$$

Solution:

(i) We will prove the following chain of implications:

$$\mu_j \rightarrow \mu \text{ narrowly} \implies a) \implies b) \implies c) \implies \mu_j \rightarrow \mu \text{ narrowly}.$$

$\mu_j \rightarrow \mu$ narrowly $\implies a)$. Let $\mu_j \rightarrow \mu$ narrowly and take A an open set. Notice that the function $\mathbb{1}_A$ is lower semicontinuous since A is open. Thus, using Exercise 3.1

$$\liminf_{j \rightarrow \infty} \mu_j(A) = \liminf_{j \rightarrow \infty} \int \mathbb{1}_A d\mu_j \geq \int \mathbb{1}_A d\mu = \mu(A).$$

$a) \implies b)$. Now assume $a)$ holds true and take C a closed set. Then $A := \mathbb{R}^n \setminus C$ is open, therefore, by $a)$:

$$\begin{aligned} \limsup_{j \rightarrow \infty} \mu_j(C) &= \limsup_{j \rightarrow \infty} (1 - \mu_j(A)) = 1 - \liminf_{j \rightarrow \infty} \mu_j(A) \\ &\leq 1 - \mu(A) = \mu(C). \end{aligned}$$

$b) \implies c)$. Assume $b)$ holds true. With the same argument as above we can easily show that $a)$ holds true as well. Let $E \subset \mathbb{R}^n$ be a set so that $\mu(\partial E) = 0$. We call $A := E \setminus \partial E$

and $C := \overline{E}$, and note that A is open and C is closed. Then, using a), b) and $\mu(\partial E) = 0$ we obtain the following chain of inequalities:

$$\begin{aligned} \limsup_{j \rightarrow \infty} \mu_j(E) &\leq \limsup_{j \rightarrow \infty} \mu_j(C) \leq \mu(C) = \mu(E) = \\ &= \mu(A) \leq \liminf_{j \rightarrow \infty} \mu_j(A) \leq \liminf_{j \rightarrow \infty} \mu_j(E). \end{aligned}$$

Hence the previous inequalities must be all equalities proving that $\mu_j(E) \rightarrow \mu(E)$, as desired.

c) $\implies \mu_j \rightharpoonup \mu$ narrowly. Assume that c) holds true and let $\varphi \in C_b(\mathbb{R}^n)$ be a nonnegative bounded continuous function. Observe that for almost every $t > 0$, the set $E_t := \{\varphi > t\}$ is such that $\mu(\partial E_t) = 0$. Hence, by c):

$$\lim_{j \rightarrow \infty} \mu_j(E_t) = \mu(E_t) \quad \text{for almost every } t > 0.$$

Therefore, using the layer cake formula, together with the dominated convergence theorem we get

$$\int \varphi d\mu = \int_0^{\max \varphi} \mu(E_t) dt = \lim_{j \rightarrow \infty} \int_0^{\max \varphi} \mu_j(E_t) dt = \lim_{j \rightarrow \infty} \int \varphi d\mu_j.$$

This proves that $\mu_j \rightharpoonup \mu$ narrowly.

- (ii) Let $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ be a sequence of points such that $|x_j| \rightarrow \infty$ as $j \rightarrow \infty$. For any $j \in \mathbb{N}$, we define $\mu_j = \delta_{x_j}$. Now for any $\varphi \in C_c(\mathbb{R}^n)$,

$$\lim_{j \rightarrow \infty} \int \varphi d\mu_j = \lim_{j \rightarrow \infty} \varphi(x_j) = 0 \quad \text{since } |x_j| \rightarrow \infty.$$

Hence, $\mu_j \xrightarrow{*} \mu$, where μ is the null measure. Now taking $A = \mathbb{R}^n$ we get

$$1 = \liminf_{j \rightarrow \infty} \mu_j(\mathbb{R}^n) > \mu(\mathbb{R}^n) = 0.$$

Exercise 3.4. Let $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R})$ be a sequence of probability measures with $\mu_n \rightharpoonup \mu$ narrowly. Define $F_n(x) := \mu_n((-\infty, x])$, $F(x) := \mu((-\infty, x])$.

- (i) Prove that μ is a probability measure.

- (ii) Prove that

$$\limsup_n F_n(x) \leq F(x) \quad \text{for every } x \in \mathbb{R}.$$

- (iii) Prove that

$$\lim_n F_n(x) = F(x) \quad \text{for every } x \in \mathbb{R} \text{ at which } F \text{ is continuous.}$$

- (iv) Give an example of a sequence of measures $\mu_n \rightharpoonup \mu$ narrowly and an $x \in \mathbb{R}$ for which

$$\limsup_n F_n(x) < F(x).$$

Solution:

- (i) To prove that μ is a probability measure, use the definition of narrow convergence with the test function $f \equiv 1$.
- (ii) We have the following

$$\begin{aligned}\limsup_n F_n(x) &= \limsup_n (1 - \mu_n((x, \infty))) \\ &= 1 - \liminf_n \mu_n((x, \infty)) \\ &\leq 1 - \mu((x, \infty)) = F(x),\end{aligned}$$

which gives the thesis.

- (iii) Let us fix a point $x \in \mathbb{R}$ and $\delta > 0$: we take a continuous non increasing function f such that $f(t) \equiv 1$ for $t \leq x$ and $f(t) \equiv 0$ for every $t \geq x + \delta$, then we have

$$F(x + \delta) \geq \int_{\mathbb{R}} f(x) d\mu(x) = \lim_n \int_{\mathbb{R}} f(x) d\mu_n(x) \geq \limsup_n F_n(x).$$

Now consider a non increasing function f such that $f(t) \equiv 1$ for $t \leq x - \delta$ and $f(t) \equiv 0$ for every $t \geq x$, then we have

$$F(x - \delta) \leq \int_{\mathbb{R}} f(x) d\mu(x) = \lim_n \int_{\mathbb{R}} f(x) d\mu_n(x) \leq \liminf_n F_n(x),$$

the thesis follows from the continuity of F in x .

- (iv) For the example consider $\mu_n = \delta_{1/n}$, $\mu = \delta_0$ and $x = 0$.

Exercise 3.5.

- (i) Find a sequence of functions $f_n : [0, 1] \rightarrow [0, 1]$ such that $(f_n)_\#(\mathcal{L}^1 \llcorner [0, 1]) = (\mathcal{L}^1 \llcorner [0, 1])$ but f_n weakly converge to $1/2$.
- (ii) What is the weak limit of $(\text{id}, f_n)_\# \mathcal{L}^1 \llcorner [0, 1]$?
- (iii) (★) Can these functions be taken C^1 ?

Hint: For (i), use piecewise affine oscillating functions.

Solution:

- (i) Define $\phi : \mathbb{R} \rightarrow \mathbb{R}$ on $[0, 1]$ by

$$\phi(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2]; \\ 2(1 - x) & \text{if } x \in (1/2, 1], \end{cases}$$

and extend it to $\mathbb{R}$ by 1-periodicity. Define $f_n: [0, 1] \rightarrow [0, 1]$ by

$$f_n(x) = \phi(nx).$$

From the Riemann-Lebesgue theorem, it is clear that f_n converges weakly to $1/2$. In order to prove $(f_n)_\#(\mathcal{L}|_{[0,1]}) = (\mathcal{L}|_{[0,1]})$ we need to show that

$$(\mathcal{L}|_{[0,1]})(f_n^{-1}(A)) = (\mathcal{L}|_{[0,1]})(A) \text{ for any } A \subseteq \mathbb{R} \text{ Borel set.} \quad (1)$$

Since the Borel σ -algebra is generated by sets of the form $A = (a, \infty)$ it suffices to show the previous equality for sets of this form. First we have

$$\phi^{-1}((a, \infty)) = \bigcup_{m \in \mathbb{Z}} (m + a/2, m + 1 - a/2) \quad \text{if } a \in [0, 1].$$

If $a \geq 1$, then $\phi^{-1}((a, \infty)) = \emptyset$ and if $a < 0$, then $\phi^{-1}((a, \infty)) = \mathbb{R}$. From this we deduce that

$$f_n^{-1}((a, \infty)) = \bigcup_{m=0}^{n-1} \left(\frac{m}{n} + \frac{a}{2n}, \frac{m+1}{n} - \frac{a}{2n} \right) \quad \text{if } a \in [0, 1].$$

In addition, $f_n^{-1}((a, \infty)) = \emptyset$ if $a \geq 1$ and $f_n^{-1}((a, \infty)) = [0, 1]$ if $a < 0$. Thus the equation (1) is satisfied if $a \geq 1$ or $a < 0$. Now considering the case when $a \in [0, 1)$, we have

$$(\mathcal{L}|_{[0,1]})(f_n^{-1}((a, \infty))) = (\mathcal{L}|_{[0,1]}) \left(\bigcup_{m=0}^{n-1} \left(\frac{m}{n} + \frac{a}{2n}, \frac{m+1}{n} - \frac{a}{2n} \right) \right) = \sum_{m=0}^{n-1} \frac{1-a}{n} = 1-a.$$

which implies (1). In conclusion, since (1) holds for any set of the form $A = (a, \infty)$ and the family of these sets generate the Borel σ -algebra, (1) holds for any Borel set which proves that

$$(f_n)_\#(\mathcal{L}|_{[0,1]}) = (\mathcal{L}|_{[0,1]}).$$

(ii) Let $\varphi \in C_c(\mathbb{R} \times \mathbb{R})$. We will show

$$\int \varphi(x, y) d(\text{id}, f_n)_\# \mathcal{L}|_{[0,1]} \rightarrow \int \varphi(x, y) d\mathcal{L}^2|_{[0,1]^2} \quad \text{as } n \rightarrow \infty.$$

Fix any $\epsilon > 0$. Since φ is uniformly continuous there is $\delta > 0$ such that if $|x - y| < \delta$, then $|\varphi(x) - \varphi(y)| < \epsilon$. Let $n \in \mathbb{N}$ be large enough so that $2/n < \delta$. Now we will consider a grid covering $[0, 1]^2$ composed by $2n^2$ rectangles of the form

$$R_{ij} = \left[\frac{i}{2n}, \frac{i+1}{2n} \right] \times \left[\frac{j}{n}, \frac{j+1}{n} \right] \quad i = 0, \dots, 2n-1, j = 0, \dots, n-1.$$

Notice that all R_{ij} have a diameter smaller than δ . Now for any $k = 0, \dots, 2n^2 - 1$, denote i_k, j_k the unique integers such that

$$\left(\frac{2k+1}{4n^2}, f_n \left(\frac{2k+1}{4n^2} \right) \right) \in R_{i_k j_k}$$

Then, using the fact that φ is uniformly continuous, we get

$$\left| \int_{\frac{k}{2n^2}}^{\frac{k+1}{2n^2}} \varphi(x, f_n(x)) dx - \frac{1}{2n^2} \varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right) \right| < \frac{1}{2n^2} \epsilon \quad \forall k = 1, \dots, 2n^2 - 1$$

and

$$\left| \int_{R_{i_k j_k}} \varphi(x, y) dx dy - \frac{1}{2n^2} \varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right) \right| < \frac{1}{2n^2} \epsilon \quad \forall k = 1, \dots, 2n^2 - 1.$$

Therefore, we get

$$\begin{aligned} & \left| \int \varphi(x, y) d(\text{id}, f_n)_\# \mathcal{L}|_{[0,1]} - \frac{1}{2n^2} \sum_{k=1}^{2n^2-1} \varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right) \right| \\ & \leq \frac{1}{2n^2} \sum_{k=1}^{2n^2-1} \left| \int_{\frac{k}{2n^2}}^{\frac{k+1}{2n^2}} \varphi(x, f_n(x)) dx - \varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right) \right| < \epsilon \end{aligned}$$

and

$$\begin{aligned} & \left| \int \varphi(x, y) d\mathcal{L}|_{[0,1]^2} - \frac{1}{2n^2} \sum_{k=1}^{2n^2-1} \varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right) \right| \\ & \leq \frac{1}{2n^2} \sum_{k=1}^{2n^2-1} \left| \int_{R_{i_k j_k}} \varphi(x, y) dx dy - \varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right) \right| < \epsilon. \end{aligned}$$

This proves that

$$\left| \int \varphi(x, y) d(\text{id}, f_n)_\# \mathcal{L}|_{[0,1]} - \int \varphi(x, y) d\mathcal{L}^2|_{[0,1]^2} \right| < 2\epsilon$$

when n is large enough. We conclude that $(\text{id}, f_n)_\# \mathcal{L}|_{[0,1]}$ converges weakly to $\mathcal{L}^2|_{[0,1]^2}$.

- (iii) Assume $\varphi \in C^1(\mathbb{R})$ such that $(\varphi)_\#(\mathcal{L}|_{[0,1]}) = \mathcal{L}|_{[0,1]}$. For a contradiction, assume there is $y \in (0, 1)$ such that $\varphi'(y) = 0$. For any $\epsilon > 0$, there is $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - y| < \delta$ we have $|\varphi'(x)| < \epsilon$. Thus, for all x such that $|x - y| < \delta$, we get

$$|\varphi(x) - \varphi(y)| < \epsilon|x - y| < \epsilon\delta$$

Then

$$(x - \delta, x + \delta) \subseteq \varphi^{-1}((\varphi(x) - \epsilon\delta, \varphi(x) + \epsilon\delta)),$$

so that

$$2\epsilon\delta = (\varphi)_\#(\mathcal{L}|_{[0,1]})(\varphi(x) - \epsilon\delta, \varphi(x) + \epsilon\delta) = (\mathcal{L}|_{[0,1]})(\varphi^{-1}(\varphi(x) - \epsilon\delta, \varphi(x) + \epsilon\delta)) \geq 2\delta$$

which yields a contradiction when taking $\epsilon < 1$. This proves that φ' never vanishes on $(0, 1)$ so that φ is either strictly increasing or strictly decreasing. Assume φ is strictly increasing

(the other case is similar). Then for any $a, b \in (0, 1)$, we have

$$b - a = (\varphi)_\#(\mathcal{L}|_{[0,1]}((a, b))) = \mathcal{L}|_{[0,1]}((\varphi^{-1}(a), \varphi^{-1}(b))) = \varphi^{-1}(b) - \varphi^{-1}(a).$$

Thus, for any $x, y \in (0, 1)$, $\varphi(x) - \varphi(y) = x - y$. We conclude $\varphi(x) = x$ on $(0, 1)$. If we assume φ is strictly decreasing, then $\varphi(x) = 1 - x$ on $(0, 1)$. We conclude by noting that any sequence of such functions cannot converge weakly to $1/2$.

Exercise 3.6 (♣). Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ be a probability measure. We say that a sequence of borel functions $T_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$ converge in μ -measure to $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ if¹

$$\lim_{j \rightarrow \infty} \mu(\{x \in \mathbb{R}^n : |T_j(x) - T(x)| > \epsilon\}) = 0 \quad \text{for every } \epsilon > 0.$$

Denoting by $\pi_j := (id, T_j)_\# \mu$, $\pi := (id, T)_\# \mu \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$, prove the following equivalence:

$$\pi_j \xrightarrow{*} \pi \quad \Longleftrightarrow \quad T_j \text{ converges to } T \text{ in } \mu\text{-measure}$$

Solution: Assume first that T_j converges to T in μ -measure. We wish to prove that $(id, T_j)_\# \mu$ weakly-* converge to $(id, T)_\# \mu$. Let $\varphi \in C_c(\mathbb{R}^n \times \mathbb{R}^n)$. Then, from each subsequence $j_k \nearrow \infty$, we can extract a further subsequence $j_{k_\ell} \nearrow \infty$ for which $T_{j_{k_\ell}}$ converges to T pointwise μ -almost everywhere, and so, by the change of variables formula and the dominated convergence theorem we obtain

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi d(id, T_{j_{k_\ell}})_\# \mu = \int_{\mathbb{R}^n} \varphi(x, T_{j_{k_\ell}}(x)) d\mu \xrightarrow{\ell \rightarrow \infty} \int_{\mathbb{R}^n} \varphi(x, T(x)) d\mu = \int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi d(id, T)_\# \mu,$$

which gives the desired result.

Let us now assume that $\pi_j = (id, T_j)_\# \mu$ weakly-* converge to $\pi = (id, T)_\# \mu$ (and hence narrowly, as all these measures have the same total mass), and prove that T_j converges to T in μ -measure. Fix $\epsilon > 0$. By Lusin's Theorem, there exists a continuous map $\tilde{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\mu(\{\tilde{T} \neq T\}) < \epsilon$. Let us then consider the continuous bounded test function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined as

$$\varphi(x, y) := \min\{|y - \tilde{T}(x)|, 1\}.$$

By the change of variables formula and the narrow convergence,

$$\begin{aligned} \int_{\mathbb{R}^n} \min\{|T_j(x) - \tilde{T}(x)|, 1\} d\mu &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi d\pi_j \rightarrow \int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi d\pi \\ &= \int_{\mathbb{R}^n} \min\{|T(x) - \tilde{T}(x)|, 1\} d\mu \leq \mu(\{T \neq \tilde{T}\}) < \epsilon. \end{aligned}$$

¹By standard measure theory arguments one can easily prove that whenever T_j converges to T in μ -measure, there is a subsequence $j_k \nearrow \infty$ such that T_{j_k} converge to T pointwise μ -almost everywhere.

Thus if j is large enough we have

$$\int_{\mathbb{R}^n} \min\{|T_j(x) - T(x)|, 1\} d\mu \leq \int_{\mathbb{R}^n} \min\{|T_j(x) - \tilde{T}(x)|, 1\} d\mu + \int_{\mathbb{R}^n} \min\{|T(x) - \tilde{T}(x)|, 1\} d\mu < 2\epsilon.$$

Finally, given any $\delta > 0$, using Markov's inequality, we get, for j large enough:

$$\mu(\{|T_j - T| > \delta\}) \leq \frac{1}{\delta} \int_{\mathbb{R}^n} \min\{|T_j(x) - T(x)|, 1\} d\mu \leq \frac{2\epsilon}{\delta},$$

and the thesis follows from the arbitrariness of ϵ .