Serie 13

Optimal transport, Fall semester

EPFL, Mathematics section, Dr. Xavier Fernández-Real

Exercise 13.1. Let $S_0, S_1 \subset \mathbb{R}^2$ be the two segments

$$S_0 := (-1,1) \times \{0\}, \qquad S_1 := \{0\} \times (-1,1).$$

Consider the two probability measures in the plane $\mu_0, \mu_1 \in \mathscr{P}(\mathbb{R}^2)$ whose mass is concentrated uniformly on the two segments S_0 and S_1, respectively:

$$\mu_0 := \mathcal{H}^1 \llcorner S_0, \qquad \mu_1 := \mathcal{H}^1 \llcorner S_1.$$

Here $\mathcal{H}$ denotes the 1-dimensional Haussdorf measure. Find a Wasserstein geodesic between μ_0 and μ_1. How many are there?

Solution: We first prove that all couplings between μ_0 and μ_1 have the same cost, and so they are all optimal. In fact, any $\gamma \in \Gamma(\mu_0, \mu_1)$ is concentrated on the set $S_0 \times S_1 \subset \mathbb{R}^2 \times \mathbb{R}^2$. Hence, by the Pythagorean Theorem, $|x-y|^2 = |x|^2 + |y|^2$ for γ-almost every $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$, and

$$\begin{aligned}\int_{\mathbb{R}^2 \times \mathbb{R}^2} |x-y|^2 d\gamma(x,y) &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} (|x|^2 + |y|^2) d\gamma(x,y) \\ &= \int_{\mathbb{R}^2} |x|^2 d\mu_0(x) + \int_{\mathbb{R}^2} |y|^2 d\mu_1(y) = \frac{4}{3}.\end{aligned}$$

Given any $\gamma \in \Gamma(\mu_0, \mu_1)$, a Wasserstein geodesic between μ_0 and μ_1 is then given by the interpolation

$$\mu_t^\gamma := [(1-t)x + ty]_{\#}\gamma \qquad t \in [0,1].$$

In particular, there are infinitely many Wasserstein geodesics connecting μ_0 to μ_1.

Exercise 13.2. Recall the Benamou-Brenier formula (see section 4.1 in Figalli-Glaudo): given two probability measures $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, then it holds that

$$W_2^2(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} |v_t|^2 d\rho_t \, dt : \partial_t \rho_t + \operatorname{div}(v_t \rho_t) = 0, \rho_0 = \mu_0, \rho_1 = \mu_1 \right\}.$$

Suppose that μ_t for $t \in [0,1]$ is a curve attaining the minimum, and suppose that $\mu_t = (X_t)_{\#}\mu_0$, for some smooth vector field X_t. Prove that $\ddot{X}_t \equiv 0$ μ_0-a.e. for a.e. $t \in (0,1)$.

Solution: Let us start by observing that, if $\rho_t = (Y_t)_\# \mu_0$ with

$$\begin{cases} \dot{Y}_t &= w_t \circ Y_t \\ Y_0 &= \mathrm{id}, \end{cases}$$

that is, $\partial_t \rho_t + \mathrm{div}(w_t \rho_t) = 0$, then we have

$$\int_{\mathbb{R}^d} |w_t|^2 \rho_t = \int_{\mathbb{R}^d} |w_t \circ Y_t|^2 \mu_0 = \int_{\mathbb{R}^d} \left|\dot{Y}_t\right|^2 \mu_0.$$

Let us consider the vector field $Y_{t,\varepsilon} := X_t + \varepsilon Z_t$, where Z_t is a smooth vector field, compactly supported in time. In particular, $Z_0 = Z_1 = 0$. Let us consider $\rho_{t,\varepsilon} := (Y_{t,\varepsilon})_\# \mu_0$, so that $\rho_{0,\varepsilon} = \mu_0$ and $\rho_{1,\varepsilon} = \mu_1$.

Then, by minimality, we have that

$$\int_0^1 \int_{\mathbb{R}^d} \left|\dot{X}_t\right|^2 d\mu_0\, dt \leq \int_0^1 \int_{\mathbb{R}^d} \left|\dot{Y}_{t,\varepsilon}\right|^2 d\mu_0\, dt = \int_0^1 \int_{\mathbb{R}^d} \left|\dot{X}_t\right|^2 d\mu_0\, dt + 2\varepsilon \int_0^1 \int_{\mathbb{R}^d} \dot{X}_t \cdot \dot{Z}_t d\mu_0\, dt + o(\varepsilon).$$

In particular, by taking $-Z_t$ instead of Z_t we deduce, letting $\varepsilon \downarrow 0$,

$$\int_0^1 \int_{\mathbb{R}^d} \dot{X}_t \cdot \dot{Z}_t d\mu_0\, dt = 0.$$

By Fubini, integrating by parts in time first, and using that $Z_0 = Z_1 = 0$,

$$0 = \int_0^1 \int_{\mathbb{R}^d} \dot{X}_t \cdot \dot{Z}_t d\mu_0\, dt = \int_{\mathbb{R}^d} \int_0^1 \dot{X}_t \cdot \dot{Z}_t dt\, d\mu_0 = -\int_{\mathbb{R}^d} \int_0^1 \ddot{X}_t \cdot Z_t\, dt\, d\mu_0.$$

From the arbitrariness of Z_t, we deduce the desired result.

Exercise 13.3. Let $\mu_0 = \rho_0 \mathscr{L}^d, \mu_1 = \rho_1 \mathscr{L}^d \in \mathcal{P}(\mathbb{T}^d)$ be two probability measures on the d-dimensional torus such that $\rho_0, \rho_1 \geq c > 0$ everywhere. Let $u : \mathbb{T}^d \to \mathbb{R}$ be a solution of the Poisson equation $-\Delta u = \rho_1 - \rho_0$. Show that

$$W_2(\mu_0, \mu_1) \leq c^{-1/2} \|\nabla u\|_{L^2}.$$

Hint: Use the Benamou-Brenier formula, which is valid also on the torus.

Solution: We will use the Benamou-Brenier formula. Let ρ_t, v_t be defined as

$$\rho_t := (1-t)\rho_0 + t\rho_1 \qquad v_t := \frac{\nabla u}{\rho_t} \qquad \forall t \in [0,1].$$

Since $-\Delta u = \rho_1 - \rho_0$, ρ_t solves the continuity equation with velocity field ρ:

$$\partial_t \rho_t + \mathrm{div}(\rho_t v_t) = 0.$$

Moreover, since $\rho_0, \rho_1 \geq c > 0$ by hypothesis, we have

$$\rho_t \geq c \qquad \forall t \in [0,1].$$

Therefore, by the Benamou-Brenier formula,

$$\begin{aligned} W_2^2(\mu_0, \mu_1) &\leq \int_0^1 \int_{\mathbb{T}^d} |v_t|^2 \rho_t dx dt \\ &= \int_0^1 \int_{\mathbb{T}^d} \frac{|\nabla u|^2}{\rho_t} dx dt \\ &\leq c^{-1} \|\nabla u\|_{L^2}^2, \end{aligned}$$

which is the desired inequality after taking the square root of both sides.

Exercise 13.4. Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set, and $V_1 : \Omega \to \mathbb{R}$, $V_2 : \Omega \to \mathbb{R}$ be functions which are lower semicontinuous and bounded from below. Show that the functionals

$$\mathbb{V}_1(\mu) = \int_\Omega V_1 \, d\mu \qquad \mathbb{V}_2(\mu) = \int_\Omega V_2(x,y) \, d\mu(x) \, d\mu(y)$$

are lower semicontinuous with respect to W_2-convergence.

Solution: Since Ω is bounded, convergence in W_2 and narrow convergence are equivalent. We saw in class that the integral with respect to μ of a lower semicontinuous function bounded below is lower semicontinuous in the space of probability measures equiped with narrow convergence. Therefore $\mathbb{V}_1$ is W_2-lower semicontinuous. In order to prove that $\mathbb{V}_2$ is W_2-lower semicontinuous is then enough to show that the narrow convergence $\mu_j \rightharpoonup \mu$ in $\mathscr{P}(\Omega)$ implies the narrow convergence $\mu_j \times \mu_j \rightharpoonup \mu \times \mu$ in $\mathscr{P}(\Omega \times \Omega)$. This is a direct consequence of the definition of narrow convergence. By a density argument, it is sufficient to test narrow convergence in the product space with functions of the type $f(x,y) = \varphi(x)\psi(y)$, with $\varphi, \psi : \Omega \to \mathbb{R}$ continuous and bounded. For these test functions we have

$$\int_{\Omega\times\Omega} f d(\mu_j \times \mu_j) = \int_\Omega \varphi(x) d\mu_j(x) \int_\Omega \psi(y) d\mu_j(y) \to \int_\Omega \varphi(x) d\mu(x) \int_\Omega \psi(y) d\mu(y) = \int_{\Omega\times\Omega} f d(\mu \times \mu),$$

and the proof is concluded.

Exercise 13.5. Show that the functional $\mathscr{F}$ given by

$$\mathscr{F}(\rho) := \int_{\mathbb{R}^d} (\rho + |x|^2) \rho dx$$

is W_2-convex, and compute the evolution equation of its Wasserstein gradient flow.

Solution: We write $\mathscr{F}(\rho)$ as the sum of two terms: an internal energy, and a potential energy,

$$\mathscr{F}(\rho) = \int_{\mathbb{R}^d} \rho^2 dx + \int_{\mathbb{R}^d} |x|^2 \rho dx.$$

The internal energy term, corresponding to $U(\rho) = \rho^2$, is W_2-convex because $s \mapsto s^d U(s^{-d}) = s^{-d}$ is non-increasing and convex (McCann criterion). On the other hand, the potential energy term, corresponding to $V(x) = |x|^2$, is W_2-convex because the potential V is convex. As a consequence, their sum $\mathscr{F}(\rho)$ is W_2-convex.

The gradient flow is formally given by the equation

$$\partial_t \rho_t = \operatorname{div}\left(\rho_t \nabla \frac{\delta \mathscr{F}}{\delta \mu}(\rho_t \mathscr{L}^d)\right),$$

where $\frac{\delta \mathscr{F}}{\delta \mu}(\rho_t \mathscr{L}^d) : \mathbb{R}^d \to \mathbb{R}$ is the L^2-variation of the functional $\mathscr{F}$ computed at $\rho_t \mathscr{L}^d$. In our case, we have

$$\frac{\delta \mathscr{F}}{\delta \mu}(\rho_t \mathscr{L}^d)(x) = U'(\rho_t(x)) + V(x) = 2\rho_t(x) + V(x).$$

Therefore, the gradient flow reads as

$$\partial_t \rho_t = \operatorname{div}(2\rho_t \nabla \rho_t) + \operatorname{div}(\rho_t \nabla V).$$