Numerical Analysis and Computational Mathematics

Fall Semester 2025 – CSE Section

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Approximation of functions and data

Exercise I (MATLAB, tutorial)

For any set of (x, y)-pairs $\{(x_i, y_i)\}_{i=0}^{n}$ such that $x_i \neq x_j$ when $i \neq j$, we can compute and evaluate the *interpolating polynomial* or a *least-squares approximating polynomial* by using the MATLAB commands `polyfit` and `polyval`.

Example 1: Let us consider the data set

i	0	1	2	3	4
x_i	0	0.25	0.5	0.75	1
y_i	3.38	3.86	3.85	3.59	3.49

The coefficients of the interpolating polynomial of degree $n = 4$ are computed using the following commands:

```
x_nodes = [0:0.25:1];
y_nodes = [3.38 3.86 3.85 3.59 3.49];
P = polyfit( x_nodes, y_nodes, 4 )    % Coefficients of the L. polynomial
%  P = 1.8133   -0.1600   -4.5933    3.0500    3.3800
```

where `P` is a vector containing the coefficients of the polynomial $\Pi_4(x) = 1.8133x^4 - 0.16x^3 - 4.5933x^2 + 3.05x + 3.38$. Given any vector of points `x_values`, we correspondingly evaluate the polynomial using the command `P_values = polyval( P, x_values )`, e.g.:

```
% polynomial value in a single point
x_value = 0.4;
P_value = polyval( P, x_value )     % value of the polynomial at point x_value
%  P_value =
%      3.9012

% polynomial values in multiple points
x_values = linspace( 0, 1, 1001);  % 1001 equispaced points between 0 and 1
P_values = polyval( P, x_values ); % values of the polynomial at points x_values
```

```
plot( x_values, P_values )          % plot the interpolating polynomial
```

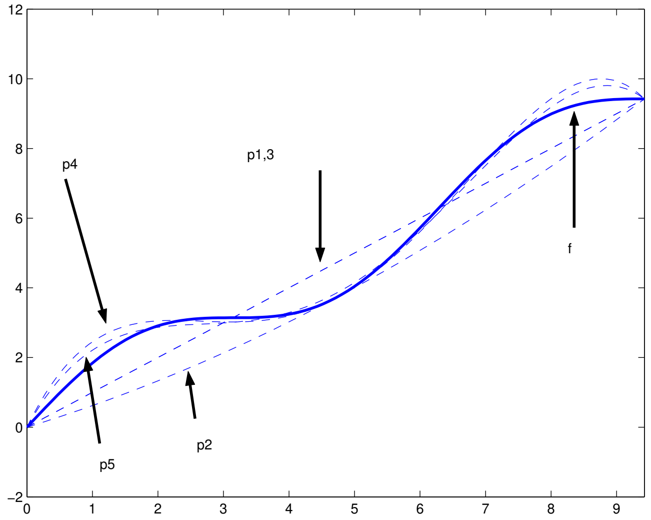
Example 2: To compute the interpolating polynomial of degree n of an arbitrary continuous function $f(x)$ we have to define a set of $n+1$ data points $\{(x_i, f(x_i))\}_{i=0}^{n}$, with x_i distinct nodes. For instance, for $f(x) = \cos(x)$, $n = 4$, and $n+1 = 5$ nodes uniformly distributed over $I = [a, b] = [0, 1]$, we execute the following commands:

```
f = @(x) cos(x); a=0; b=1;
n = 4;
x_nodes = linspace( a, b, n +1 );
y_nodes = f(x_nodes);
P = polyfit( x_nodes, y_nodes, n )
%  P =
%       0.0362    0.0063   -0.5025    0.0003    1.0000
```

Remark: If the length of `x_nodes` (and `y_nodes`) is larger than $n+1$ (where n is the degree of desired interpolating polynomial) the command `polyfit(x_nodes, y_nodes, n)` returns the *least-squares approximating polynomial* of degree n.

Example 3: Let us interpolate the function $f(x) = \sin(x) + x$ at $n+1 = 2, 3, \ldots, 6$ uniformly distributed nodes in $I = [0, 3\pi]$ by means of interpolating polynomials $\Pi_n f(x)$ of degree n; we use the following commands:

```
f = @(x) sin(x) + x; a=0; b=3*pi;
x_values = linspace(a,b,1001);
plot( x_values, f(x_values), 'b' );
hold on
for n = 1:5
    x_nodes = linspace( a, b, n+1 );
    P = polyfit( x_nodes, f(x_nodes), n );
    plot( x_values, polyval( P, x_values ), '--b' )
end
```



Exercise II (MATLAB)
Consider the function $f(x) = \sin(x)$ on the interval $I = [a, b] = [0, 3\pi]$.

a) By distributing the $n + 1$ nodes uniformly over the interval I, compute the interpolating polynomial $\Pi_n f$ of the function $f(x)$ for $n = 1, \ldots, 7$, where n is the degree of the polynomial. Compare the results obtained with the analytical expression of $f(x)$. (*Hint*: plot $\Pi_n f$ and $f(x)$ on the same figure using at least 1001 evaluation points.)

b) Compute the true errors $e_n(f)$:

$$e_n(f) := \max_{x\in I} |E_n f(x)|, \qquad \text{with } E_n f(x) := f(x) - \Pi_n f(x),$$

for $n = 1, \ldots, 7$. Plot them in a figure as a function of n, and comment the result. (*Hint*: instead of the maximum over I, compute the maximum over a fine grid in I.)

c) We recall that, in the case of interpolating polynomials of functions $f(x) \in C^{n+1}(I)$ at uniformly spaced nodes, the true errors $e_n(f)$ can be bounded as

$$e_n(f) \leq \widetilde{e}_n(f), \qquad \text{where } \widetilde{e}_n(f) := \frac{1}{4(n+1)} \left(\frac{b-a}{n}\right)^{n+1} \max_{x\in I} \left|f^{(n+1)}(x)\right|.$$

Plot the error estimators $\widetilde{e}_n(f)$ vs. the degree n. Compare the plot with that of point b), and verify the validity of the bound above.

Exercise III (Theoretical)
Consider the function $f(x) = -x^3 + 3x^2 - 2$ with $x \in I = [0, 2]$.

a) Compute the quadratic interpolating polynomial $\Pi_2 f(x)$ at the nodes $x_0 = 0$, $x_1 = \frac{1}{2}$, and $x_2 = 2$. Provide the expressions of the Lagrange characteristic functions.

b) Repeat point a) with $x_1 = 1$ and motivate the result obtained.

c) Compute the cubic interpolating polynomial $\Pi_3 f(x)$ at the nodes $x_0 = 0$, $x_1 = e^{-\sqrt{2}}$, $x_2 = 3e^{-\sqrt{1/2}}$ and $x_3 = 2$.

Exercise IV (MATLAB)
Consider the Runge function $f(x) = \dfrac{1}{1+x^2}$ over the interval $I = [a, b] = [-5, 5]$.

a) By distributing the nodes uniformly over I, use MATLAB to compute and plot the interpolating polynomials $\Pi_n f(x)$ for degrees $n = 2, 4, 8$, and 12. Compare the results with the plot of the function $f(x)$.

b) Compute the "true" errors $e_n(f) := \max_{x\in I} |f(x) - \Pi_n f(x)|$ for $n = 2, 4, 8$, and 12, and plot them vs. the degree n. Is the error decreasing when n increases? Motivate the answer.

c) Repeat point a) by using the Chebyshev-Gauss-Lobatto distribution of the nodes in I. In this case the $n + 1$ nodes x_i are obtained as:

$$x_i = \frac{a+b}{2} + \frac{b-a}{2}\hat{x}_i, \qquad \text{where } \hat{x}_i = -\cos\left(\pi\frac{i}{n}\right), \quad i = 0, \ldots, n.$$

For $n = 8$, compare the interpolating polynomial with the corresponding polynomial obtained at point a).

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d) Repeat point b) by using the Chebyshev-Gauss-Lobatto distribution of the nodes in I obtained at point c) and motivate the results obtained.

Exercise V (Theoretical)
Consider the function $f(x) = \sin\left(\frac{x}{3}\right)$ over the interval $I = [a, b] = [0, 1]$.

a) Let $\Pi_n f(x)$ be the interpolating polynomial of $f(x)$ at $n + 1$ uniformly spaced nodes $\{a = x_0, x_1, \ldots, x_n = b\}$. Provide an estimate $\widetilde{e}_n(f)$ of the interpolation error $e_n(f) := \max_{x\in I} |f(x) - \Pi_n f(x)|$ over the interval I, as a function of the degree n. Study the limit of $\widetilde{e}_n(f)$ for $n \to \infty$.

b) Find the minimum number of uniformly spaced nodes over I that guarantee that $e_n(f) < 10^{-4}$. (*Hint*: try increasing values of n until the condition is satisfied).

c) Consider the polynomial interpolation at the Chebyshev-Gauss-Lobatto nodes. Compute the first four Chebyshev-Gauss-Lobatto nodes x_0, x_1, x_2, x_3 over the interval I.

d) We define the nodal polynomial $\omega_n(x) := \prod_{i=0}^{n}(x - x_i)$, with $\{x_i\}_{i=0}^{n}$ the nodes used for the interpolation. We set $n = 3$ and we use the Chebyshev-Gauss-Lobatto nodes obtained at point c). We show in Figure 1 the plot of $|\omega_3|$ over I. Use the plot to estimate the interpolation error $e_n(f)$ corresponding to the Chebyshev-Gauss-Lobatto nodes over I.

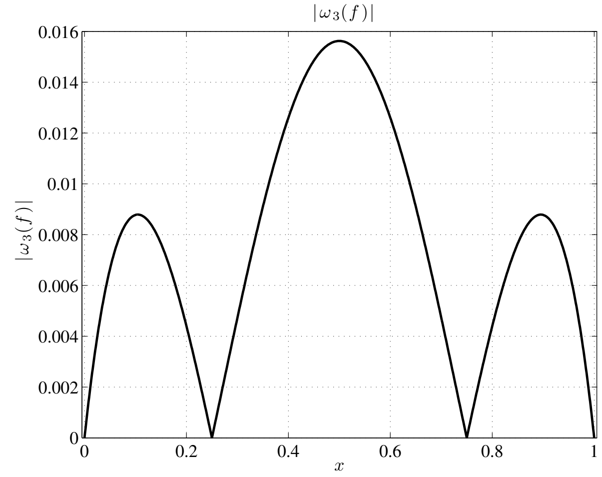

Figure 1: Function $|\omega_3(x)|$ in the interval I.