EPFL

Numerical Analysis and Computational Mathematics

Fall Semester 2025 – CSE Section

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Nonlinear equations: fixed point iterations

Exercise I (MATLAB)

We aim at solving the equation $f(x) = 0$ in $[a, b]$, with $f(x) = x - \cos(x)$, $a = -\pi/3$ and $b = \pi/3$. The function $f(x)$ has one zero, which we denote by $\alpha \in [a, b]$, where $\alpha \simeq 0.739085133215161\ldots$. We consider the fixed point iterations algorithm, with iteration function $\phi(x) = \cos(x)$. The generic iteration of the algorithm is:

$$x^{(k+1)} = \phi(x^{(k)}), \quad \text{for all } k \geq 0, \tag{1}$$

for a given initial value $x^{(0)}$.

a) With the help of the file `fixed_point_iterations_template.m`, implement the fixed point iterations method in a MATLAB function, to be saved in file `fixed_point_iterations.m`. The layout of the function is the following:

```
function [xvect, nit] = fixed_point_iterations( phi, x0, tol, nmax )
% FIXED_POINT_ITERATIONS Finds a fixed point of a scalar function.
%   [XVECT] = FIXED_POINT_ITERATIONS(PHI,X0,TOL,NMAX) finds a fixed point of
%   the iteration function PHI using the fixed point iterations method and
%   returns a vector XVECT containing the successive approximations of the
%   fixed point (iterates).
%   PHI accepts a real scalar input x and returns a real scalar value;
%   PHI can also be an inline object. X0 is the initial guess.
%   TOL is the tolerance on error allowed and NMAX the maximum number of iterations.
%   The stopping criterion based on the difference of successive iterates is used.
%   If the search fails a warning message is displayed.
%
%   [XVECT,NIT] = FIXED_POINT_ITERATIONS(PHI,X0,TOL,NMAX) also returns the
%   number of iterations  NIT.
%   Note: the length of the vectors is equal to ( NIT + 1 ).
%

return
```

By using the function `fixed_point_iterations`, verify that α is a fixed point of $\phi(x)$ for a few choices of $x^{(0)} \in [a, b]$, with $tol = 10^{-6}$ and $k_{max} = 1500$.

b) Report the number of iterations k_c required for the convergence of the algorithm and the error $e^{(k_c)} = |x^{(k_c)} - \alpha|$.

c) For the convergent cases, plot $e^{(k)} = |x^{(k)} - \alpha|$ vs. k. What can be said about the order of convergence of the algorithm?

Exercise II (Theoretical and MATLAB)

We consider the equation $f(x) = 0$ in $[a, b]$, with $f(x) = x/2 - \sin(x) + \pi/6 - \sqrt{3}/2$, $a = -\pi/2$ and $b = \pi$. The function $f(x)$ has two zeros, $\alpha_1 \in I_1 = [-\pi/2, 0]$ and $\alpha_2 \in I_2 = [\pi/2, \pi]$. In particular, $\alpha_1 \simeq -1.047197598567\ldots$ and $\alpha_2 \simeq 2.246005589297\ldots$. We use fixed point iterations to find the zeros α_1 and α_2, with the iteration function $\phi(x) = \sin(x) + x/2 - \pi/6 + \sqrt{3}/2$.

a) Verify that α_2 is a fixed point of $\phi(x)$ and establish if the method is:

   (a) *globally convergent* in I_2, i.e. the method converges for all $x^{(0)} \in I_2$. (*Hint*: you can use MATLAB to plot and evaluate the function $\phi(x)$ and its derivative $\phi'(x)$.)

   (b) *locally convergent*, i.e. the method converges to α_2 if $x^{(0)}$ is sufficiently close to α_2.

b) By using the function `fixed_point_iterations`, verify the answer given to point a) for two different choices of $x^{(0)} \in I_2$, with $tol = 10^{-6}$ and $k_{max} = 1500$. Report the number of iterations (k_c) required for the convergence of the algorithm and the error $e^{(k_c)} = |x^{(k_c)} - \alpha_2|$. In addition, for the convergent cases, plot $e^{(k)} = |x^{(k)} - \alpha_2|$ vs. k. Moreover, plot the ratio $a^{(k)} := (x^{(k+1)} - \alpha_2)/(x^{(k)} - \alpha_2)$ vs. k. Discuss the results obtained.

c) Repeat point a) for $\alpha_1 \in I_1$.

d) Repeat point b) to verify the convergence to $\alpha_1 \in I_1$. Try using $x^{(0)} = -1.1$ and $x^{(0)} = -0.9$.

e) Consider $\alpha_2 \in I_2$ and the fixed point algorithm (1). Prove that there exists a positive constant $0 \leq C < 1$ such that, for all $x^{(0)} \in I_2$,

$$|x^{(k+1)} - \alpha_2| \leq C|x^{(k)} - \alpha_2|, \quad \text{for all } k \geq 0 \tag{2}$$

(*hint*: use the mean value theorem). Compute the value of C explicitly.

f) Following point e), prove that, given (2), we can write

$$|x^{(k)} - \alpha_2| \leq C^k|x^{(0)} - \alpha_2|, \quad \text{for all } k \geq 0.$$

Then, assuming that α_2 is unknown, use this result to estimate the minimum number of iterations k_{min} such that the error $|x^{(k_{min})} - \alpha_2|$ is smaller than $tol = 2^{-20}$ for all choices of $x^{(0)} \in I_2$.

g) Is the stopping criterion based on the difference between successive approximations $x^{(n)}$ satisfactory when using fixed point iterations to find α_1 and α_2? Motivate and verify your answer from a theoretical point of view and by using the function `fixed_point_iterations` (set $tol = 10^{-6}$ and $k_{max} = 1500$, and run your code for the cases of the fixed points α_1 and α_2). Specifically, set $x^{(0)} = -1.1$ for finding α_1 and $x^{(0)} = \pi/2$ for finding α_2.

Exercise III (Theoretical)

Consider the function $f(x) = e^x + 3\sqrt{x} - 2$, where $x \in I = [0.02, 0.2]$.

a) Show that f has a zero $\alpha \in I$ and that it is unique.

b) Consider the fixed point iterations algorithm with the following two iteration functions:

$$\phi_1(x) = \log(2 - 3\sqrt{x}) \quad \text{and} \quad \phi_2(x) = \frac{(2 - e^x)^2}{9}.$$

Which iteration function would you use to find $\alpha \in I$? Why? (*Hint*: plot them in MATLAB.)

c) To verify your answer to the previous point, represent qualitatively the behavior of the fixed point iterations for $\phi_1(x)$ and $\phi_2(x)$, starting from $x^{(0)} = 0.05$.

Exercise IV (OPTIONAL, Theoretical)

Let $\alpha \in (a, b)$ be a zero of the function $f : [a, b] \to \mathbb{R}$, where $f \in C^q([a, b])$, with $q = \max(m, 2)$ for some $m \geq 1$. We say that α has multiplicity $m \geq 1$ if $f(\alpha) = f'(\alpha) = \ldots = f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$.

a) Write the Newton method as a fixed point iterations algorithm and define the corresponding iteration function $\phi_N(x)$.

b) Following point a), prove that $\phi_N'(\alpha) = 1 - 1/m$ for all $m \geq 1$, where α is a zero with multiplicity m. (*Hint*: rewrite the function $f(x)$ as $f(x) = (x - \alpha)^m g(x)$ in a neighborhood of α, with $g(\alpha) \neq 0$).

c) From points a) and b), deduce the convergence properties of the Newton method, when interpreted as a fixed point iterations algorithm. Specifically, discuss the cases of a zero α of multiplicity $m = 1$ and $m > 1$.

d) The modified Newton method can also be cast as a fixed point iterations algorithm. Similarly to points a) and b), find the corresponding iteration function $\phi_{N_m}(\alpha)$ and its derivative $\phi_{N_m}'(\alpha)$ at a zero α of multiplicity $m \geq 1$. As in point c), discuss the convergence properties of the modified Newton method.

e) Interpreting the Newton method as a fixed point iterations algorithm, discuss the quality of the stopping criterion based on the difference of successive approximations $x^{(n)}$ in the cases of a zero α of multiplicity $m = 1$ and $m > 1$. What do you expect for $m = 100$?