EPFL

Numerical Analysis and Computational Mathematics

Fall Semester 2025 – CSE Section

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Nonlinear equations: bisection and Newton methods

Exercise I (MATLAB)

a) Write a MATLAB function implementing the bisection method (in a file called `bisection.m`) by following the layout found in the file `bisection_template.m`. The layout of the function reads:

```
function [xvect,esterrvect,resvect,nit] = bisection(fun,a,b,tol,nmax)
% BISECTION Find a zero of a nonlinear scalar function inside an interval.
%   XVECT=BISECTION(FUN,A,B,TOL,NMAX) finds a zero of the continuous
%   function FUN in the interval [A,B] using the bisection method and returns
%   a vector XVECT containing the successive approximations of the zero (iterates).
%   FUN accepts real scalar input x and returns a real scalar value;
%   FUN can also be an inline object.
%   TOL is the tolerance on error allowed and NMAX the maximum number of iterations.
%   If the search fails an error message is displayed.
%
%   [XVECT,ESTERRVECT,RESVECT,NIT]=BISECTION(FUN,...) also returns the vector
%   ESTERRVECT of error estimators for each iterate, the vector RESVECT of residual
%   evaluations for each iterate, and NIT the number of iterations.
%   Note: the length of the vectors is equal to ( NIT + 1 ).
%

return
```

where `fun` is an inline function defined outside `bisection.m`. Note that, inside `bisection.m`, it is possible to evaluate `fun` at a point `x` using the command `fx = fun(x)`. (*Hint*: follow the pseudocodes in Algorithm 2.1 and Algorithm 2.2 in the notes).

Verify the implementation of the bisection method by approximating the zero $\alpha \in (a, b)$ of the function $f(x) = \sin(2x) - 1 + x$, for $a = -1$ and $b = 3$, with a graphical visualization of the approximate solution `xvect`. Set the tolerance for the stopping criterion to $tol = 10^{-1}$ and the maximum number of iterations to $n_{max} = 100$. You may use as template the script `bisection_example.m`:

```
fun = @(x) sin(2*x) - 1 + x; % inline function
a = -1; b = 3;
xv = linspace(a,b,2001); % xv=[a:(b-a)/2000:b]
plot(xv, fun(xv)); grid on; % plot of the function f
tol = 1e-4; nmax = 100;
[xvect,esterrvect,resvect,nit] = bisection(fun,a,b,tol,nmax);
x_nit = xvect(end); % final iterate (approximated zero); NOTE: length(xvect)=nit+1
fx_nit = resvect(end); % final residual fun(x nit); NOTE: length(resvect)=nit+1
hold on
plot(xvect,resvect,'*k',x_nit,fx_nit,'or'); % iterates in black, the final in red
```

b) Repeat for decreasing values of *tol* $= 10^{-2}, 10^{-3}$ and 10^{-4}, and verify that reducing the tolerance results in a convergent sequence of approximated solutions (*hint*: use the magnifying glass tool to zoom in close to the zero).

c) **(Optional)** Improve the function `bisection.m` to make it more robust. If $f(a)$ and $f(b)$ have the same sign, an error message should be displayed. If one of the endpoints of the interval (a, b) given as input is a zero of the function, it should be returned as the solution without performing any iteration.

Exercise II (Theoretical and MATLAB)

Consider the problem of finding the zero $\alpha \in (0, 2)$ of the function $f(x) = (1 - x)\sin(4x) + 1/6$.

a) Plot the function $f(x)$ in MATLAB for $x \in [0, 2]$. Does it satisfy the conditions required to apply the bisection method? Motivate the answer.

b) Estimate the minimum number of iterations n_{min} required by the bisection method to reach an approximation $x^{(n_{min})}$ of the zero α yielding an error $e^{(n_{min})} := |x^{(n_{min})} - \alpha| < \epsilon$, with $\epsilon = 10^{-6}$.

c) Following point b), use the `bisection.m` function to perform the first 20 iterations of the bisection method, when applied to the function $f(x)$ for $x \in [0, 2]$. What are the values of the residuals $r^{(n)} := f(x^{(n)})$ corresponding to iterations $n = 19$ and $n = 20$?

d) We say that the residuals $r^{(n)} := f(x^{(n)})$ converge monotonically with respect to n if $|r^{(n)}| \geq |r^{(n+1)}|$ for all $n \geq 0$. In general, this property is desirable for an iterative method because the solution after taking another iteration can never be worse than the previous one. Is the convergence of the residuals monotonic for the function $f(x)$ considered here? (*Hint*: following point c), plot in semilogarithmic scale the absolute residuals $|r^{(n)}|$ vs. n.)

e) An iterative method converges *linearly* (convergence of order 1) if:

$$\lim_{n\to\infty} \frac{|x^{(n+1)} - \alpha|}{|x^{(n)} - \alpha|} = \mu, \qquad \text{for some } 0 < \mu < 1, \tag{1}$$

where μ is the asymptotic convergence factor. Let us assume that, after 20 iterations, the method has found a solution that is very close to the zero, so that $\alpha \simeq x^{(20)}$ (otherwise stated, we take $x^{(20)}$ as approximate value of α in our convergence analysis). Plot the sequence $a_n := |x^{(n+1)} - x^{(20)}|/|x^{(n)} - x^{(20)}|$ for the first 18 approximate zeros $x^{(n)}$. Is it possible to verify the order of convergence from this graph by using definition (1)?

f) As an alternative, consider the sequence of error estimators $\tilde{e}^{(n)} := |I^{(n+1)}| = (b-a)/2^{n+1}$ (for $0 \leq n \leq 18$). Does it converge linearly, i.e.:

$$\lim_{n\to\infty} \frac{|\tilde{e}^{(n+1)}|}{|\tilde{e}^{(n)}|} = \nu, \qquad \text{for some } 0 < \nu < 1?$$

What does this tell you about the convergence of the method?

Exercise III (Theoretical and MATLAB)

Consider the function $f(x) = x^3 - 2x - 5$ on the interval $(1, 3)$. Since $f(x)$ is a polynomial, it is continuous and continuously differentiable. We can observe that:

$$f(1) = -6 < 0 \quad \text{and} \quad f(3) = 16 > 0,$$

thus, it has at least one zero $\alpha \in (1, 3)$.

a) Verify the uniqueness of the zero $\alpha \in (1, 3)$.

b) Write the Newton scheme for the function $f(x)$.

c) Compute the first three Newton iterations starting from the initial guess $x^{(0)} = 1.5$.