Exercises for Statistical analysis of network data – Sheet 11

1. We give the solutions of (i)-(xiv) only for hypergraph (a), except for when no example of the concept of interest can be found in it. In that case, we will make explicit to which graph we refer to. The others are for your further happy training for graph marathon!

   (i) Order = 6, $n_{\text{edges}} = 11$.

   (ii) Edge $\{2, 6\}$ in edge $\{2, 4, 6\}$ (top circle); edges $\{3, 4\}$ and $\{5, 6\}$ in edge $\{3, 4, 5, 6\}$ (right-hand side circle); edge $\{1, 5\}$ in edge $\{1, 3, 5\}$ (bottom circle); edges $\{3, 4\}$ and $\{1, 2\}$ in edge $\{1, 2, 3, 4\}$ (left-hand side circle).

   (iii) Node groups that are encircled, that is, $\{2, 4, 6\}$ (top circle), $\{3, 4, 5, 6\}$ (right-hand side circle), $\{1, 3, 5\}$ (bottom circle), and $\{1, 2, 3, 4\}$ (left-hand side circle).

   (iv) Graph (a) is not a simple graph: it has included edges as seen above as well as multiple edges.

   (v) Not adjacent: 1 and 6, 2 and 5. All other pairs are adjacent.

   (vi) Vertex 1: $d_1 = 5$, neighbourhood: $\{2, 3, 4, 5\}$
   Vertex 2: $d_2 = 5$, neighbourhood: $\{1, 3, 4, 6\}$
   Vertex 3: $d_3 = 5$, neighbourhood: $\{1, 2, 4, 5, 6\}$
   Vertex 4: $d_4 = 5$, neighbourhood: $\{1, 2, 3, 5, 6\}$
   Vertex 5: $d_5 = 4$, neighbourhood: $\{1, 3, 4, 6\}$
   Vertex 6: $d_6 = 4$, neighbourhood: $\{2, 3, 4, 5\}$

   (vii)

   (viii) Edge list: $\{1, 2\}$, $\{1, 3\}$, $\{1, 5\}$, $\{2, 4\}$, $\{2, 6\}$, $\{3, 4\}$, $\{5, 6\}$, $\{2, 4, 6\}$, $\{1, 3, 5\}$, $\{3, 4, 5, 6\}$, and $\{1, 2, 3, 4\}$. The last two have cardinality 4, the two before them have cardinality 3, the others have cardinality 2.

   (ix) $\Delta(\mathcal{H}_a) = 5$.

   (x) Graph (a) is not regular (not all degrees are equal).

   (xi) Graph (a) is not uniform (not all edges have the same cardinality).

   (xii) The rank of graph (a) is $r(\mathcal{H}_a) = 4$.

   (xiii) Isolated nodes cannot be found in either of the graphs (a), (b) and (c). Graph (c) contains a pendant node, the other two does not.

   (xiv) No singleton or empty edges can be found in any of the three graphs.

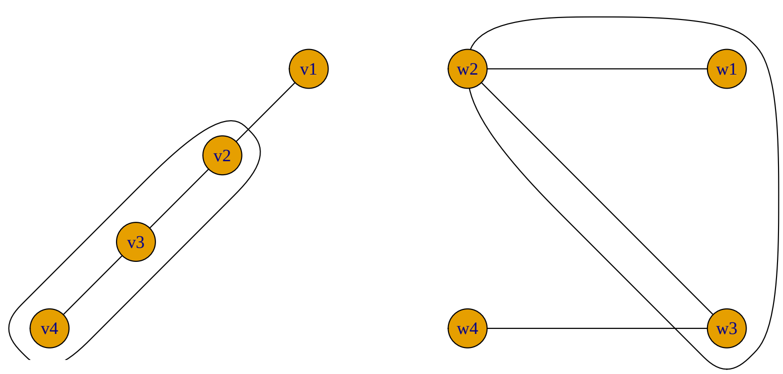

Figure 1:

   (xv) See Figure 1. for the two graphs. The edge set of the first is $e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_4, e_4 = v_2v_3v_4$. The edge set of the second is $f_1 = w_1w_2, f_2 = w_2w_3, f_3 = w_3w_4, f_4 = w_1w_2w_3$. An isomorphism between the two is $\Phi(v_1) = w_4, \Phi(v_2) = w_3, \Phi(v_3) = w_2, \Phi(v_4) = w_1$. Since $\Phi(e_1) = w_3w_4 = f_3, \Phi(e_2) = w_2w_3 = f_2, \Phi(e_3) = w_1w_2 = f_1, \Phi(e_4) = w_1w_2w_3 = f_4$, and this constitutes a bijection, the two hypergraphs are isomorph.

(xvi) The incidence matrix of the graph on the left-hand side of Figure1, using the same edge labeling as in point (xv), is

$$I = \begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline v_1 & 1 & 0 & 0 & 0 \\ v_2 & 1 & 1 & 0 & 1 \\ v_3 & 0 & 1 & 1 & 1 \\ v_4 & 0 & 0 & 1 & 1 \end{array}$$

Its dual is the graph whose incidence matrix is the transpose of the above, that is,

$$I = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline e_1 & 1 & 1 & 0 & 0 \\ e_2 & 0 & 1 & 1 & 0 \\ e_3 & 0 & 0 & 1 & 1 \\ e_4 & 0 & 1 & 1 & 1 \end{array}$$

The resulting graph is shown in Figure 2. The former edges now become vertices (see labels on vertices). There is an edge now with a single node (corresponding to the former node v_1, which was a pendant.

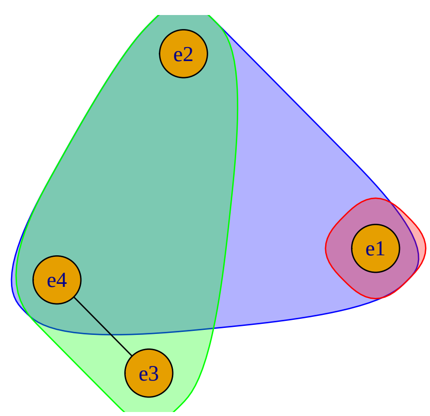

Figure 2: