Exercises for Statistical analysis of network data – Sheet 13

1. Assume you observe two networks $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$ defined on the same vertex set V, where $|V| = n$. $\mathcal{G}^{(1)}$ is an Erdős–Rényi network with edge probability $0 < p_1 < 1$ where the edge variables are collected in an adjacency matrix $A^{(1)}$, and $\mathcal{G}^{(2)}$ is represented by the adjacency matrix $A^{(2)}$ for $0 < p_2 < 1$ generated according to:

$$A_{ij}^{(2)} \mid A^{(1)} = \text{Bernoulli}(p_2 + (p_1 - p_2)A_{ij}^{(1)}), \quad 1 \leq i < j \leq n. \tag{1}$$

(i) Determine the expectation of the edge variables, as well as their covariance. What is the marginal distribution of the set of edge variables $\{A_{ij}^{(2)}\}$ if you consider them in isolation?

(ii) What is the marginal distribution of the edge variables $\{A_{ij}^{(1)}\}$ if you consider them in isolation?

Now define the edge predictor

$$\hat{A}_{ij}^{(2)} = \begin{cases} 1 & \text{if} \quad p_2 + (p_1 - p_2)A_{ij}^{(1)} > 1/2 \\ 0 & \text{if} \quad p_2 + (p_1 - p_2)A_{ij}^{(1)} \leq 1/2 \end{cases}.$$

Calculate the expectation of $(\hat{A}_{ij}^{(2)} - A_{ij}^{(2)})^2$ using the law of iterated expectation.