MIDTERM EXAM

Probabilité Avancée

Midterm Autumn Semester 2024

November 5 2024

Length of exam : 1h 45 minutes

Permitted : no pages of personnel notes are allowed.

No calculation machines are allowed.
Please start each question on a fresh page.

Attempt all the questions

Any dishonesty will be severely sanctioned !

Write first your full name and section :

Name :______________________________ **Given name :** ______________________________

Departement : _____________________

Exercice	**Points**
1	
2	
3	
Total points :	

1)(i) State the Monotone convergence Theorem

(ii) What is convergence in probability ? Give a weak law of large numbers result. Points according to generality.

(iii) State Fatou's Lemma.

(iv)Given two probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$, what is the product sigma field $\mathcal{F}_1 \times \mathcal{F}_2$?

(v) Is the uncountable intersection of σ fields (of subsets of a common Ω) necessarily a σ field ?

(vi) For measure space $(\Omega, \mathcal{F})$, P and Q are two probabilities. If $P(A) = Q(A)\ \forall\ A \in \mathcal{A}$ where $\sigma(\mathcal{A}) = \mathcal{F}$ must $P = Q$? Justify.

1)contd

2a) Let $X_1, X_2, \ldots,$ be independant identically distributed r.v.s taking values in $\mathbb{N} \cup \{-1\} \backslash \{0,1\}$, so that

$$\forall n \geq 2 \quad P(X_1 = n) = \frac{C}{n^2 log^2(n)},$$

$$P(X_1 = -1) = 1 - \sum_n \frac{C}{n^2 log^2(n)};$$

where C is chosen so that $E(X_1) = 0$. We define $S_n = \sum_1^n X_i$. Show that

$$\frac{S_n}{n} \rightarrow 0,$$

in probability. Is it possible to extend this to a.s. convergence? Using the truncation at level $\frac{n}{log^{3/2}(n)}$ or otherwise, show that

$$\frac{S_n}{n/log(n)} \rightarrow -C,$$

in probability. Is it possible to extend this to convergence a.s.? Justify. (You may use approximations of sums via corresponding integrals without proof.)

2 contd

3) a) For X a positive random variable with $E(X) = 1$ on space $(\Omega, \mathcal{F}, P)$, show carefully that

$$A \to \int_A X dP$$

is a probability measure on $(\Omega, \mathcal{F})$.

b) Consider $E \subset \mathbb{R}^d$ (not necessarily measurable) and let $\mathcal{B}_E$ be the sigma field of subsets of E generated by sets of the form $O \cap E$ for open sets O in $\mathbb{R}^d$. Show that

$$\mathcal{B}_E = \{E \cap B\} \text{ for } B \text{ Borelian .}$$

If P and Q are probability measures on $(E, \mathcal{B}_E)$ so that for each open set O $P(O \cap E) = Q(O \cap E)$, is it necessarily the case that $P = Q$?

3contd