1 a) A λ system Λ (for a given set Ω).
is a collection of subsets of Ω so that

(i) $\Omega \in \Lambda$.

(ii) $A, B \in \Lambda,\ A \subseteq B \implies B \setminus A \in \Lambda$

(iii) $A_n \in \Lambda\ \forall n \quad A_n \uparrow A \to A \in \Lambda$

b) Consider $X > 0$, a r.v. with $E(X) = \infty$

Take $X_n = \frac{X}{n}$ so $X_n \geq X_{n+1}\ \forall n$ ~~then~~ and

$\forall n \quad E(X_n) = E\left(\frac{X}{n}\right) = \frac{1}{n}E(X) = \infty$. ~~So~~ we have

$X_n = \frac{X}{n} \to 0$ so

$\lim E(X_n) = \infty \neq E \lim X_n = 0.$

c) We wish to use the generalization of BC II:

$\star$ If $V_n = \sum_{j=1}^{n} I_{A_j}$ where events A_j are pairwise indep

and $\sum_{j=1}^{\infty} P(A_j) = \infty$ then

$$\frac{V_n}{E(V_n)} = \frac{V_n}{\sum_{1}^{n} P(A_j)} \xrightarrow{a.s.} 1.$$

We have $S_n = \sum_{1}^{n} I_{A_j}$ with $A_j = \left\{ X_j > X_{j+1} + \frac{\log c}{3} \right\}$

(note difference with printed exam)

Now $\forall j$ X_j, X_{j+1} are indep $\text{Exp}(1)$ r.v.s so

$m(x_j, x_{j+1}) = \lambda x_j \otimes \lambda x_{j+1}$ is law on $\mathbb{R}^2$ with

density $e^{-x}e^{-y}$ so $P(A_j) = P\left((X_j, X_{j+1}) \in \left\{(x,y) : x > y + \frac{\log c}{3}\right\}\right)$

$$= \int_0^\infty dy\, e^{-y} \int_{y + \frac{\log c}{3}}^{\infty} e^{-x}\, dx = \int_0^\infty e^{-y} e^{-y - \log c/3}\, dy$$

$$= \frac{1}{c^{1/3}} \int_0^\infty e^{-2y}\, dy = \frac{1}{2c^{1/3}}$$

(AND SO $\sum_{j=1}^{n} P(A_j) = \sum_{j=1}^{n} \frac{1}{2j^{1/3}} \approx \frac{3}{4} n^{2/3} \to \infty$)

UNFORTUNATELY THE EVENTS $\{A_j\}_{j=1}^{\infty}$ ARE NOT PAIRWISE

INDEP. eg $P(A_i \cap A_{i+1}) \leq P(A_i \cap X_{i+1} > \frac{\log(i+1)}{3})$

$\leq P(X_i > \frac{\log i}{3} + \frac{\log(i+1)}{3}, X_{i+1} > \log\frac{(i+1)}{3})$

$\overset{\text{INDEP}}{=} P(X_i > \frac{\log i}{3} + \frac{\log(i+1)}{3}) P(X_{i+1} > \log\frac{(i+1)}{3})$

$= \cancel{\frac{1}{2i^{1/3}}} \frac{1}{(i+1)^{1/3}} \frac{1}{i^{1/3}} \frac{1}{(i+1)^{1/3}} << P(A_i) P(A_{i+1})$

A $\overset{\text{FOR CLARITY}}{\text{TRICK}}$ IS TO USE $S_n = \sum_{\substack{i \leq n \\ i \text{ EVEN}}} I_{A_i} + \sum_{\substack{i \leq n \\ i \text{ ODD}}} I_{A_i}$

THE EVENTS $A_2, A_4, A_6 \ldots$ ARE INDEP. SAME $A_1, A_3, A_5 \ldots$

SO BY (*) $\frac{\sum_{i \text{ EVEN}}^{n} I_{A_i}}{\sum_{\substack{i \leq n \\ i \text{ EVEN}}} P(A_i)} \xrightarrow{\text{a.s.}} 1$ AND $\frac{\sum_{\substack{i \leq n \\ i \text{ ODD}}}^{n} I_{A_i}}{\sum_{i \text{ ODD}}^{n} P(A_i)} \xrightarrow{\text{a.s.}} 1$

SO $\frac{S_n}{E(S_n)} = \frac{\sum_{\substack{i \text{ EVEN} \\ i \leq n}} I_{A_i} + \sum_{\substack{i \text{ ODD} \\ i \leq n}} I_{A_i}}{\sum_{\substack{i \leq n \\ i \text{ EVEN}}} P(A_i) + \sum_{\substack{i \leq n \\ i \text{ ODD}}} P(A_i)} \to 1.$

(SO $\frac{S_n}{3/4\, n^{2/3}} \xrightarrow{\text{a.s.}} 1$)

d) $\overline{\lim} A_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{m=n}^{\infty} A_m \right) \equiv \bigcap_{n=1}^{\infty} B_n$

NOW $B_n \downarrow \overline{\lim} A_n$ SO BY CONTINUITY OF PROBS

$P(\overline{\lim} A_n) = \lim_{n\to\infty} P(B_n) = \overline{\lim}\, P(B_n)$

$\geq \overline{\lim}\, P(A_n)$ SINCE $\forall n$ $P(B_n) \geq P(A_n)$

e) FIX $c>0$ AND CONSIDER $A_n = \{n^{1/2} X_n \le c\}$

$= \{X_n \le c/n^{1/2}\}$. FOR $n^{1/2} \ge c$ $P(A_n) = \frac{c}{n^{1/2}}$.

SO BY BC II $P(\overline{\lim} A_n) = 1$ (AS $\sum c/n^{1/2} = \infty$)

BUT ON $\overline{\lim} A_n$ WE HAVE $\liminf n^{1/2} X_n \le c$.

SO a.s. $\underline{\lim} \sqrt{n} X_n = 0$ AS c IS ARBITRARY SO AND

$\sqrt{n} X_n \ge 0$ $\forall n$.

f) YES $P = Q$

SUPPOSE $\forall$ $0 \le b-a, d-c \le 1$

$[a,b] \cap [c,d] = [a \vee c, b \wedge d]$ IS A CLOSED

INTERVAL OF LENGTH ≤ 1. SO

$\Pi = \{[a,b] : b-a \le 1\}$ IS A Π-SYSTEM.

BY DYNKINS Π-λ LEMMA AND OUR HYPOTHESIS ON P, Q

$P(A) = Q(A)$ $\forall A \in \sigma(\Pi)$.

IT REMAINS TO SHOW

SHOWING $\sigma(\Pi) = \mathcal{B}$:

FOR ANY $a<b \in \mathbb{R}$ $[a,b] = \bigcup_{n=1}^{\infty} [a, a+n \wedge b]$.

SO $\{[a,b]\ a<b\} \subseteq \sigma(\Pi)$.

ALSO $]a,b[= \bigcup_{n=1}^{\infty} [a+\frac{1}{n}, b-\frac{1}{n}]$ $\forall a<b$

SO $\{]a,b[: a<b\} \subseteq \sigma(\Pi)$. BUT ANY OPEN SET O

$= \bigcup I_i$ I_i IS OPEN INTERVAL, O SO

$O \subseteq \sigma(\Pi)$ $\forall O$ OPEN. SO

$\mathcal{B} \subseteq \sigma(\Pi)$ BY DEFN OF BOREL σ-FIELD

2 a) $X_n \xrightarrow{P} X$ IFF $\forall \epsilon > 0$ $P(|X_n - X| \geq \epsilon) \xrightarrow{n\to\infty} 0$.

b) WE USE TRUNCATION. FOR n FIXED WE PUT (FOR $1 \leq i \leq n$)

$X_i' = X_i \wedge c_n$ FOR c_n TO BE CHOSEN.

WE WRITE $S_n' = \sum_1^n X_i'$. WE WANT c_n TO BE SUCH THAT $P(S_n \neq S_n') \xrightarrow{n\to\infty} 0$. SUBJECT TO THIS WE WANT c_n "AS SMALL AS POSSIBLE".

$$P(S_n \neq S_n') \leq P(\exists\, i \leq n;\ X_i \neq X_i').$$

$$\overset{\text{UNION BOUND}}{\leq} \sum_{i=1}^n P(X_i \neq X_i') = n P(X_1 \neq X_1') \ (\text{iid}).$$

$= n \frac{\log_2(c_n)+1}{c_n}$ IF $c_n \geq 1$ (WHICH WILL BE THE CASE)

WE ~~HATAKE~~ TAKE $c_n = n \log_2 n (\log_2 \log_2 n)$ THEN.

$$n P(X_1 \neq X_1') = P(X_1 > c_n) = n \frac{\log_2 c_n + 1}{c_n}$$

$$= \frac{\log_2(n) + \log_2\log_2(n) + \log_2\log_2\log_2(n) + 1}{n \log_2(n) \log_2\log_2(n)} \leq \frac{2}{\log_2\log_2 n}$$

FOR n LARGE $\xrightarrow{n\to\infty} 0$

SO TO SHOW $\frac{2S_n}{n\log_2^2 n} \xrightarrow{P} 1$ IT IS ENOUGH TO SHOW THAT

$\frac{2S_n'}{n\log_2^2 n} \xrightarrow{P} 1$.

NOW $E(S_n') = nE(X_1')$ AS X_i' $1 \leq i \leq n$ ARE iid.

$$= n\int_0^\infty P(X_1' \geq x)\,dx = n + n\int_1^{c_n} P(X_1 \geq x)\,dx$$

$$= n + n\int_1^{c_n} \frac{\log_2 x + 1}{x}\,dx = n + n\int_0^{\log_2 c_n} u+1\,du \quad \text{CofV}$$

$$= n + n \left.\frac{u^2}{2} + u\right|_0^{\log_2 c_n} = n + n\left(\frac{\log_2^2 c_n}{2} + \log_2 c_n\right)$$

$$= \cancel{n} + n\frac{\log_2^2 n}{2}(1+o(1))$$

2b) CONT

SO TO PROVE $\frac{2S_n'}{n\log_2^2 n} \xrightarrow{pr} 1$ IT IS ENOUGH TO SHOW

$$\frac{S_n'}{E(S_n')} \xrightarrow{pr} 1$$

i.e. $\forall \epsilon>0$ $P\left(\left|\frac{S_n'-E(S_n')}{E(S_n')}\right| \geq \epsilon\right) \to 0.$

WE FIX $\epsilon>0$ WE WANT TO USE

$$(*) \quad P\left(\left|\frac{S_n'-E(S_n')}{E(S_n')}\right| \geq \epsilon\right) \leq \frac{1}{\epsilon^2 E(S_n')^2} VAR(S_n').$$

$$= \frac{n}{\epsilon^2 E(S_n')^2} VAR(X_i') \text{ AS } X_i' \; 1\leq i\leq n \text{ ARE IID.}$$

$$\leq \frac{n}{\epsilon^2 E(S_n')^2} E(X_i'^2)$$

SO TO SHOW THIS TENDS TO ZERO WE NEED ONLY GET A GOOD BOUND FOR $E(X_i'^2)$

$$E(X_i'^2) = \int_0^\infty 2x P(X_i' \geq x)\,dx$$

$$= \int_0^1 2x\,dx + \int_1^{C_n} 2x \frac{\log_2(x)+1}{x}\,dx$$

$$= 1 + 2\int_1^{C_n} \log_2(x)+1\,dx \leq 1+2C_n(\log_2(C_n)+1)$$

$\leq 4 C_n \log_2 n$ FOR n LARGE $\leq 6 n \log_2^2 n \log_2\log_2 n$ FOR n LARGE

SO $\frac{n}{\epsilon^2 E(S_n')^2} E(X_i'^2) \leq \frac{n 6 \, n\log_2^2 n \log_2\log_2 n}{\epsilon^2 (n\log_2^2 n/2)^2}$ FOR n LARGE

$$= \frac{24}{\epsilon^2}\frac{n^2}{n^2}\frac{\log_2^2 n \log_2\log_2 n}{\log_2^4 n} = \frac{24}{\epsilon^2}\frac{\log_2\log_2 n}{(\log_2 n)^2} \xrightarrow{n\to\infty} 0$$

SO WE ARE DONE

(v') NO! THE CONVERGENCE DOES NOT EXTEND TO a.s. WE ARGUE BY THE ABSURD:

SUPPOSE

$$\frac{2S_n}{n\log^2 n} \xrightarrow{a.s} 1. \quad \text{THEN } \frac{2S_{n-1}}{n\log^2 n} = \frac{2S_{n-1}}{(n-1)\log^2 (n-1)} \cdot \frac{(n-1)\log^2(n-1)}{n\log^2(n)}$$

$$\to 1 \times 1 = 1.$$

SO

$$\frac{2S_n}{n\log^2 n} - \frac{2S_{n-1}}{n\log^2 n} = \frac{2(S_n - S_{n-1})}{n\log^2 n} \to 1-1=0$$

BUT

$$2\,\frac{S_n - S_{n-1}}{n\log^2 n} = \frac{2X_n}{n\log^2 n}$$

SO THIS MUST TEND TO ZERO a.s.

BUT FOR EVENT $A_n = \{X_n > n\log^2 n\}$ WE HAVE BY DISTR OF X_n (= DISTR OF X_1)

$$P(A_n) = \frac{\log(n\log^2 n)+1}{n\log^2 n} \geq \frac{\log n}{n\log^2 n} = \frac{1}{n\log(n)} \quad \text{FOR } n > 1$$

SO $\sum_{n=1}^{\infty} P(A_n) = \infty$ BY INTEGRAL TEST

SO AS X_n AND SO A_n ARE INDEPENDENT, WE MAY APPLY BC II TO CONCLUDE

$$\text{a.s} \quad \overline{\lim}\ \frac{2X_n}{n\log^2 n} \geq 2. \text{ IN PARTICULAR}$$

$$\text{a.s} \quad \frac{2X_n}{n\log^2 n} \not\to 0. \text{ CONTRADICTION, THEREFORE}$$

$$\frac{2S_n}{n\log^2 n} \stackrel{a.s}{\not\to} 0$$

3) a) THIS IS STANDARD WAY TO PROVE EVERY ELEMENT OF A σ-FIELD HAS A GIVEN PROPERTY!

(i) SHOW EVERY ELEMENT OF A CONVENIENT GENERATING COLLECTION HAS THE GIVEN PROPERTY

(ii) SHOW THAT THE SET OF SETS HAVING THE PROPERTY IS A σ-FIELD

FOR US THE PROPERTY IS $W^{-1}(A) \in \mathcal{F}$ FOR $W: \Omega \to M$ A GIVEN ORDINARY FUNCTION

OUR σ-FIELD IS $\sigma(\mathcal{A})$ WHERE $\mathcal{A}$ IS A COLLECTION OF SUBSETS OF M SATISFYING THIS PROPERTY SO (i) ABOVE IS IMMEDIATE. FOR (ii) WE MUST SHOW THAT $\mathcal{H} = \{A : W^{-1}(A) \in \mathcal{F}\}$ IS A σ-FIELD.

a) $M \in \mathcal{H}$: $W^{-1}(M) = \Omega \in \mathcal{F}$ FOR ANY $\mathcal{F}$ σ-FIELD

b) $A \in \mathcal{H} \Rightarrow A^c \in \mathcal{H}$: IF $A \in \mathcal{H} \overset{\text{DEFN}}{\Rightarrow} W^{-1}(A) \in \mathcal{F}$

$\Rightarrow W^{-1}(A)^c \in \mathcal{F}$. BUT $W^{-1}(A)^c = W^{-1}(A^c)$ SO

$W^{-1}(A^c) \in \mathcal{H} \overset{\text{DEFN}}{\Rightarrow} A^c \in \mathcal{H}$.

c) $A_n \in \mathcal{H}$ $n=1,2,\ldots \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{H}$: $\forall n\ A_n \in \mathcal{H} \Rightarrow$

$\forall n\ W^{-1}(A_n) \in \mathcal{H} \Rightarrow$ (σ-FIELDS CLOSED UNDER COUNTABLE UNION)

$\bigcup W^{-1}(A_n) \in \mathcal{H}$. BUT $\bigcup W^{-1}(A_n) = W^{-1}(\bigcup_1^{\infty} A_n)$ SO

$W^{-1}(\bigcup_1^{\infty} A_n) \in \mathcal{H}$ i.e. (DEFN OF $\mathcal{H}$) $\bigcup_1^{\infty} A_n \in \mathcal{H}$.

b(ii) TO SHOW THAT Q IS A PROB ON $(\Omega, \mathcal{F})$ WE MUST SHOW

a) $Q(\Omega) = 1$: $Q(\Omega) \overset{\text{DEF}}{=} E(X I_{\Omega}) = E(X) = 1$
(BY HYPOTHESIS ON X)

b) $Q(A) \in [0,1]\ \forall A \in \mathcal{F}$: X IS POSITIVE SO $0 \le I_A X \le X$ AND SO BY MONOTONICITY OF THE INTEGRAL

$0 \le E(I_A X) = Q(A) \le E(X) = 1$.

c) For $\{A_n\}_{n=1}^{\infty}$ disjoint events. $Q(\bigcup_1^{\infty} A_n) = \sum_{n=1}^{\infty} Q(A_n)$:

Let $A = \bigcup_1^{\infty} A_n$

$Q(A) \stackrel{\text{Def}}{=} E(I_{\bigcup A_n} X)$

$0 \le I_{\bigcup_{n=1}^{N} A_n} X \uparrow I_{\bigcup A_n} X$ as $N \to \infty$

So by monotone convergence

$Q(A) = E(I_{\bigcup A_n} X) = \lim_{N \to \infty} E(I_{\bigcup^N A_n} X)$

linearity of integral plus $I_{\bigcup A_n} = \sum_1^N I_{A_n}$ as A_i disjoint

$= \lim_{N \to \infty} \sum_{n=1}^{N} E(I_{A_n} X) \stackrel{\text{Def}}{=} \lim_{N \to \infty} \sum_1^N Q(A_n)$

$= \sum_1^{\infty} Q(A_n)$.

b(ii) Recall $\int Y(\omega) Q(d\omega) \equiv E^Q(Y)$. The expectation of Y on your prob space $(\Omega, \mathcal{F}, Q)$

Hence the "standard" approach to showing results for arbitrary bounded r.v.s Y.

a) For $Y = I_A$, $A \in \mathcal{F}$

$E^Q(Y) = Q(A) \stackrel{\text{Def}}{=} E(X I_A)$ (by defn of Q)

So result is true for special Y of this form.

b) For $Y = \sum_1^N c_i I_{A_i}$, $c_i \in \mathbb{R}$, $A_i \in \mathcal{F}$

By linearity

$E^Q(Y) = \sum_1^N c_i E^Q(I_{A_i}) \stackrel{\text{(part a)}}{=} \sum_1^N c_i E(X I_{A_i})$

linearity of E $= E(X \sum_1^N c_i I_{A_i}) = E(XY)$

c) For Y general $\exists$ Y_n simple so that for some $K < \infty$

$|Y_n| \le K$ as $\forall n$ $|Y| \le K$, $Y_n \uparrow Y$

Then $E^Q(Y) \stackrel{\text{Bounded conv.}}{=} \lim_{n \to \infty} E^Q(Y_n) \stackrel{(b)}{=} \lim_{n \to \infty} E(X Y_n) = E(XY)$

where the last equality follows from Dom. conv as $\forall n$ $(X Y_n) \le K X$ which is integrable

3) WE MAY EXTEND THIS TO POSITIVE RVS Y

BY MONOTONE CONVERGENCE: $\forall\, Y \geq 0\ \exists\ Y_n$ BOUNDED ≥ 0

S.T. $Y_n \uparrow Y$. THEN

$$E^Q(Y) \underset{M.C.}{=} \lim_{n\to\infty} E^Q(Y_n) \overset{\text{RESULT OF (ii)}}{=} \lim_{n\to\infty} E(XY_n) = E(XY)$$

BY MONOTONE CONVERGENCE AGAIN