

Problem Set 6

27.10.2025

Exercise 1.

Let $(H_t, t \in [0, T])$ be an adapted process such that $\mathbb{E}(H_t^2) < +\infty$ for all $t \in [0, T]$, and $(s, t) \mapsto \mathbb{E}(H_s H_t)$ is a continuous function on $[0, T]^2$. Show that

$$\int_0^T H_s dB_s = \lim_{n \rightarrow \infty} \sum_{i=1}^n H_{t_{i-1}^{(n)}} (B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}}) \quad \text{in } L^2,$$

where $0 = t_0^{(n)} < t_1^{(n)} < \dots < t_n^{(n)} = T$ and $\lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} (t_i^{(n)} - t_{i-1}^{(n)}) = 0$.

Exercise 2.

Let $(B_t, t \geq 0)$ be a standard Brownian motion. Define

$$Z = \int_0^1 s dB_s.$$

- (a) Is this integral well-defined?
- (b) What's the value of $\mathbb{E}(Z)$ and $\text{Var}(Z)$?
- (c) What is the law of Z ?

Exercise 3. (Polarization identity and definition of quadratic covariation)

1) Let X, Y be two square integrable random variables. Show that

$$\text{Cov}(X, Y) = \frac{1}{4} (\text{Var}(X + Y) - \text{Var}(X - Y)).$$

2) Let $(X_t), (Y_t)$ be two continuous and square integrable martingales with respect to the same filtration (\mathcal{F}_t) . We define the *quadratic variation* of (X_t) and (Y_t) as:

$$\langle X, Y \rangle_t := \frac{1}{4} (\langle X + Y \rangle_t - \langle X - Y \rangle_t), \quad t \in \mathbb{R}_+.$$

Show that the process $(X_t Y_t - \langle X, Y \rangle_t, t \in \mathbb{R}_+)$ is a martingale with respect to (\mathcal{F}_t) .