
Problem Set 5

27.10.2025

Exercise 1.

Let (X, Y) be a Gaussian vector on \mathbb{R}^2 such that $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ and $\mathbb{E}(Y^2) > 0$. Show that $\mathbb{E}(X|Y) = \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}Y$.

Exercise 2.

1) Let $(M_t, t \in \mathbb{R}_+)$ be a square integrable continuous martingale. Prove that (M_t) is an orthogonal increments process, that is, it satisfies the following property:

$$\mathbb{E}((M_t - M_s) M_s) = 0, \quad \forall t > s \geq 0.$$

Deduce that $\text{Cov}(M_t, M_s)$ is a function of $t \wedge s$.

2) Let $(X_t, t \in \mathbb{R}_+)$ be a central Gaussian process satisfying Markov property and

$$\mathbb{E}((X_t - X_s) X_s) = 0, \quad \forall t > s \geq 0.$$

Show that (X_t) is a martingale respect to its natural filtration (\mathcal{F}_t^X) .

Exercise 3.

Let $M = (M_t, t \in \mathbb{R}_+)$ be a continuous martingale such that $\mathbb{E}(M_t^2) < +\infty$, for all $t \in \mathbb{R}_+$.

(a) Using Doob's decomposition theorem to show that there is a unique continuous increasing adapted process $(\langle M \rangle_t, t \in \mathbb{R}_+)$ with $\langle M \rangle_0 = 0$, such that $(M_t^2 - \langle M \rangle_t, t \in \mathbb{R}_+)$ is a martingale. (The process $(\langle M \rangle_t)$ is called the *quadratic variation of M*).

(b) Show that for $s < t$,

$$\mathbb{E}((M_t - M_s)^2 | \mathcal{F}_s) = \mathbb{E}(M_t^2 - M_s^2 | \mathcal{F}_s) = \mathbb{E}(\langle M \rangle_t - \langle M \rangle_s | \mathcal{F}_s).$$

Exercise 4. (Stochastic integral of a simple predictable process with respect to a martingale)

Let $(M_t, t \in \mathbb{R}_+)$ be a continuous martingale such that $\mathbb{E}(M_t^2) < +\infty$, for all $t \in \mathbb{R}_+$. Given a simple predictable process $(H_t, t \in \mathbb{R}_+)$ such that

$$H_t = \sum_{i=1}^n X_i \mathbf{1}_{]t_{i-1}, t_i]}(t),$$

where $0 = t_0 < t_1 < \dots < t_n = T$ and X_i is $\mathcal{F}_{t_{i-1}}$ -measurable and bounded for all $i = 1, \dots, n$, we denote that

$$(H \cdot M)_T = \sum_{i=1}^n X_i (M_{t_i} - M_{t_{i-1}}).$$

(a) Show that $H \mapsto (H \cdot M)_T$ is linear.

(b) Show that $\mathbb{E}((H \cdot M)_T) = 0$.

(c) Let $(\langle M \rangle_t, t \in \mathbb{R}_+)$ be the quadratic variation of (M_t) . Show that

$$\mathbb{E}((H \cdot M)_T^2) = \mathbb{E} \left(\int_0^T H_s^2 d\langle M \rangle_s \right).$$