Martingale representation theorem

Let $(\mathcal{F}_t, t \in \mathbb{R}_+)$ be the natural filtration of a standard Brownian motion $(B_t, t \in \mathbb{R}_+)$:

$$\mathcal{F}_t = \sigma(B_s, s \in [0,t]).$$

Let $(M_t, t \in \mathbb{R}_+)$ be a martingale (relative to $(\mathcal{F}_t)$), such that $E(M_t^2) < \infty$, $\forall t \in \mathbb{R}_+$.

Then there is an adapted process $(H_t, t \in \mathbb{R}_+)$ such that

$$E\left(\int_0^t H_s^2 \, ds\right) < \infty, \quad \forall t \in \mathbb{R}_+,$$

and $\forall t \in \mathbb{R}_+$,

$$M_t = M_0 + \int_0^t H_s \, dB_s, \quad \text{a.s.,} \qquad (*)$$

Remarks (a) In particular, (M_t) is continuous, because it is a stochastic integral. This conclusion and property $(*)$ are specific to the natural filtration of BM.

(b) (H_s) is unique, and so is M_0 (because $M_0 = E(M_t)$): indeed,

if $M_t = M_0 + \int_0^t \tilde{H}_s \, dB_s$, then $\int_0^t (\tilde{H}_s - H_s) \, dB_s = 0$ a.s.,

so $E\left(\int_0^t (\tilde{H}_s - H_s)^2 \, ds\right) = 0$, that is,

$$\tilde{H}_s = H_s \quad ds\, dP\text{-a.e.}$$

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Sketch of proof.

Step 1. Let $X_t = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right)$. Then

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t, \quad X_0 = 1 \quad \text{(Ex. 2, hdt. set 10).}$$

If $\mu = 0$, this means that

$$X_t = 1 + \int_0^t \sigma X_u\, dB_u,$$

that is,

$$\exp\left(\sigma B_t - \frac{\sigma^2}{2}t\right) = 1 + \int_0^t \sigma \exp\left(\sigma B_u - \frac{\sigma^2}{2}u\right) dB_u,$$

therefore,

$$\exp(\sigma B_t) = \exp\left(\frac{\sigma^2}{2}t\right) + \int_0^t \sigma \exp\left(\sigma B_u + \frac{\sigma^2}{2}(t-u)\right) dB_u.$$

$= E(\exp(\sigma B_t)) + \int_0^T 1_{\{u \le t\}} \exp(\dots)\, dB_u \quad (*1)$; $e^{\frac{\sigma^2 t}{2}}$ is deterministic : this gives a representation for variables of the form $e^{\sigma B_t}$.

Step 2. Fix $T > 0$, $t_k \in [0,T]$, $k = 1, \dots, n$. Let Y be a r.v. of the form

$$Y = \sum_{k=1}^n \alpha_k \exp(\sigma_k B_{t_k}).$$

Then there is $(H_s) \in \mathcal{X}_T(B)$ such that

$$Y = E(Y) + \int_0^T H_s\, dB_s.$$

Proof. By Step 1, H_s is a linear combination of processes that appear in $(*1)$.

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Step 3. $\exp(i\theta B_t) = \exp\left(-\frac{\theta^2}{2}t\right) + \int_0^t i\theta \exp\left(-\frac{\theta^2}{2}(t-s) + i\theta B_u\right) dB_u$.

$i^2 = -1$; $\sigma = i\theta$

Formally: apply Itô's formula to $f(x) = e^{i\theta x + \frac{\theta^2}{2}t}$, gives

$$e^{i\theta B_t + \frac{\theta^2}{2}t} = 1 + \int_0^t i\theta\, e^{i\theta B_s + \frac{\theta^2}{2}s}\, dB_s \quad \text{(deterministic terms cancel out)}$$

then multiply both sides by $e^{-\frac{\theta^2}{2}t}$.

Or: apply Itô's formula separately to the real and imaginary parts, which are

$$e^{\frac{\theta^2}{2}t} \cos(\theta B_t) \quad \text{and} \quad e^{\frac{\theta^2}{2}t} \sin(\theta B_t).$$

<u>Step 4.</u> $\exp(i\theta(B_{t+s}-B_s)) = \exp(-\frac{\theta^2}{2}t) + \int_s^{s+t} i\theta \exp(-\frac{\theta^2}{2}(s+t-u)+i\theta(B_u-B_s))\,dB_u$.

<u>Proof.</u> Apply Step 3 to the BM $B^s = (B_{t+s}-B_s,\ t\in\mathbb{R}_+)$.

Notice that the stochastic integral is over $[s,s+t]$, which is the interval on which the increment of B is calculated.

$[s,\ s+t]$

From Step 3, we get $\exp(-\frac{\theta^2}{2}t) + \int_0^t i\theta \exp(-\frac{\theta^2}{2}(t-r)+i\theta(B_{r+s}-B_s))\,dB^s_r$

where $B^s_r = B_{r+s}-B_r$. Then set $u = s+r$.

Fri. 8.12.2025

<u>Step 5.</u> Suppose that X and Y are bounded r.v's such that

$$X = x_0 + \int_0^T H^1_s\,dB_s, \qquad Y = y_0 + \int_0^T H^2_s\,dB_s,$$

where $(H^1_s)\in\mathcal{H}_T(B)$ and $(H^2_s)\in\mathcal{H}_T(B)$. If $\int_0^T H^1_s H^2_s\,ds = 0$, then $(*)$

$$XY = x_0y_0 + \int_0^T (X_s H^2_s + Y_s H^1_s)\,dB_s.$$

where $X_s = E(X|\mathcal{F}_s)$, $Y_s = E(Y|\mathcal{F}_s)$.

<u>Proof.</u> Since $t\mapsto \int_0^t H^i_s\,dB_s$ is a martingale,

$$X_t = E(X|\mathcal{F}_t) = x_0 + E\left(\int_0^T H^1_s\,dB_s \,\middle|\, \mathcal{F}_t\right) = x_0 + \int_0^t H^1_s\,dB_s,$$

similarly, $Y_t = E(Y|\mathcal{F}_t) = y_0 + \int_0^t H^2_s\,dB_s$.

Use integration by parts:

$$XY = X_T Y_T = X_0Y_0 + \int_0^T X_s\,dY_s + \int_0^T Y_s\,dX_s + \langle X,Y\rangle_T$$
$$= x_0y_0 + \int_0^T (X_s H^2_s + Y_s H^1_s)\,dB_s + \underbrace{\int_0^T H^1_s H^2_s\,ds}_{=0}.$$

<u>Step 6.</u> Let $0 = t_0 < t_1 < \cdots < t_n = T$. Let

$$Z = \prod_{j=1}^n \exp(i\theta_j(B_{t_j}-B_{t_{j-1}})).$$ Then there is $H\in\mathcal{H}_T(B)$

such that $Z = E(Z) + \int_0^T H_s\,dB_s$.

<u>Proof.</u> Apply Steps 4 and 5 (condition $(*)$ holds).

Step 7. If $X_n = E(X_n) + \int_0^T H_s^n \, dB_s$, $(H_s^n) \in \mathcal{H}_T(B)$, $\forall n \in \mathbb{N}$,

and if $\lim_{n \to \infty} E((X_n - X)^2) = 0$, then there is $H \in \mathcal{H}_T(B)$

such that $$X = E(X) + \int_0^T H_s \, dB_s.$$

Proof. The hypotheses imply $E(X_n) \to E(X)$, and $(X_n - E(X_n))$ is a Cauchy sequence in $L^2(\Omega)$.

By the Itô isometry, (H_s^n) is a Cauchy sequence in $\mathcal{H}_T(B)$.

Since $\mathcal{H}_T(B)$ is a Hilbert space, $(H_s^n) \to (H_s)$ in $\mathcal{H}_T(B)$,

so $$\int_0^T H_s^n \, dB_s \xrightarrow{L^2(\Omega)} \int_0^T H_s \, dB_s .$$ (again by the isometry).

Since $X_n = E(X_n) + \int_0^T H_s^n \, dB_s$, we get $X = E(X) + \int_0^T H_s \, dB_s$. ∎

Step 8. The set of linear combinations of r.v.'s as in Step 6 is dense in $L^2(\Omega, \mathcal{F}_T, P)$.

No proof here (this uses the fact that $(\mathcal{F}_t)$ is the natural filtration of (B_t)).

Step 9. Set $X = M_T$, then use Steps 6, 7 and 8 to get

$$M_T = \underbrace{E(M_T)}_{= E(M_0) = M_0} + \int_0^T H_s \, dB_s \quad \text{with } (H_s) \in \mathcal{H}_T(B)$$

because $\mathcal{F}_0 = \{\phi, \Omega\}$ since $B_0 = 0$.

Step 10. For $t \in [0, T]$,

$$M_t = E(M_T \mid \mathcal{F}_t) = M_0 + E\left(\int_0^T H_s \, dB_s \mid \mathcal{F}_t\right)$$
$$= M_0 + \int_0^t H_s \, dB_s . \quad \blacksquare$$

End 22.05.2024