EPFL - Section MA — Theory of Stochastic Calculus
Prof. R. Dalang — Spring 2024

Final Exam — Room GR A3 32, June 21, 2024

- Calculations and mathematical arguments should be briefly justified.
- Duration of the exam : 2h.
- No documents nor electronic devices are allowed.
- There are 6 problems to solve.

Problem 1 ($\sim$ 20 minutes). On the interval $[0,1[$, consider the stochastic differential equation

$$dX_t = -\frac{X_t}{1-t}\,dt + dB_t, \quad X_0 = 0.$$

(a) Show that $X_t = (1-t)\int_0^t \frac{1}{1-s}\,dB_s$, $t \in [0,1[$, is a solution of this equation.
(b) For $s,t \in [0,1[$, calculate $E(X_s X_t)$.
(c) Set $Y_t = B_t - tB_1$, $t \in [0,1[$. Noting that (Y_t) and (X_t) are Gaussian processes, show that they have the same probability law.

Problem 2 ($\sim$ 15 minutes). Let $(B_t,\ t \geq 0)$ be a standard Brownian motion. For $t > 0$, set

$$\langle B \rangle_t^{(n)} = \sum_{j=1}^{2^n} (B_{jt2^{-n}} - B_{(j-1)t2^{-n}})^2.$$

Prove directly that $\lim_{n\to\infty} \langle B \rangle_t^{(n)} = t$ a.s.

Problem 3 ($\sim$ 20 minutes). Let $q, f : \mathbb{R} \longrightarrow \mathbb{R}$ be bounded Borel functions. Assume that the equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + q(x)u(t,x), \quad t \in \mathbb{R}_+,\ x \in \mathbb{R},$$

with initial condition $u(0,x) = f(x)$, $x \in \mathbb{R}$, has a unique bounded solution, such that $\frac{\partial u}{\partial x}$ is bounded.
Show that

$$u(t,x) = E\left(f\left(x + B_t\right) \exp\left(\int_0^t q\left(x + B_s\right)\,ds \right) \right),$$

where $(B_s,\ s \in \mathbb{R}_+)$ is a standard Brownian motion.

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Problem 4 ($\sim$ 15 minutes). Let $T > 0$, $(B_t,\ t \geq 0)$ be a standard Brownian motion and $g : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz function.

(a) Show that there is a constant $C < +\infty$ with the following property : if $(Y_t,\ t \in \mathbb{R}_+)$ and $(Z_t,\ t \in \mathbb{R}_+)$ are two continuous adapted processes, then for all $t \in [0, T]$,

$$E\left[\sup_{0\leq s\leq t}\left(\int_0^s (g(Y_r) - g(Z_r))\, dB_r\right)^2\right] \leq C\, E\left[\int_0^t (Y_r - Z_r)^2\, dr\right].$$

(b) Let $x_0 \in \mathbb{R}$. Using (a) and a Gronwall's lemma, show that the stochastic differential equation

$$dX_t = g(X_t)\, dB_t, \qquad X_0 = x_0,$$

has at most one solution. Remember to state the Gronwall's lemma that you are using.

Problem 5 ($\sim$ 15 minutes). Let $T \in \mathbb{R}_+$, $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)$ be given. Let $Y \in \mathcal{H}_T(B)$. Set

$$X_t = \exp\left(-\int_0^t Y_s\, dB_s - \frac{1}{2}\int_0^t Y_s^2\, ds\right).$$

Assume that $(X_t,\ t \in [0, T])$ is a martingale.

(a) Explain why $Q(G) = E_P[1_G\, X_T]$ defines a probability measure on $(\Omega, \mathcal{F}_T)$.

(b) Show that if $(X_t\, M_t,\ t \in [0, T])$ is a martingale under P, then $(M_t,\ t \in [0, T])$ is a martingale under Q.

Problem 6 ($\sim$ 15 minutes). Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Fix $T > 0$.

(a) For all $n \geq 1$, define $\tau_n = \inf\left\{t \in]0, T] : \int_0^t e^{2B_s^4}\, ds \geq n\right\}$. Show that for all $t > 0$, $P\{\tau_n \leq t\} > 0$.

(b) Explain how to determine a process $M_t = \int_0^t e^{B_s^4}\, dB_s$, $t \in [0, T]$, from simple predictable processes.

(c) Is (M_t) a martingale or a local martingale? Please explain your answer.