Series 9 14/11/2025

Exercise 1.

Let $(B_t, t \in \mathbb{R}_+)$ be a standard Brownian motion. We define $M_t := B_t^2 - t$ and $N_t := \exp\left(B_t - \frac{t}{2}\right)$. These two processes are martingales (see Exercise 2 Series 3). Provide the Doob decomposition of M_t^2 and N_t^2, then deduce $\langle M \rangle_t$ and $\langle N \rangle_t$.

Exercise 2. (Ornstein-Uhlenbeck Process)

Let $(B_t, t \in \mathbb{R}_+)$ be a standard Brownian motion, $a \in \mathbb{R}^*$ fixed and $X_t := \int_0^t e^{-a(t-s)}\, dB_s$, $t \in \mathbb{R}_+$.

(a) Calculate $\mathbb{E}(X_t)$ and $\mathrm{Cov}(X_s, X_t)$.

(b) Is the process (X_t) a martingale ?

(c) Prove that (X_t) satisfies the following equation :

$$X_t = -a \int_0^t X_s\, ds + B_t.$$

Exercise 3.

Let $(B_t, t \geqslant 0)$ be a standard Brownian motion. Show that the following processes are martingales

(a) $X_t = e^{\frac{t}{2}} \sin(B_t)$,

(b) $X_t = (B_t + t) e^{-B_t - \frac{t}{2}}$.

Exercise 4.

Let $(B_t, t \geqslant 0)$ be a standard Brownian motion.

(a) Let $\mu \in \mathbb{R}$. Consider the Itô process $X_t = B_t + \mu t$. What is the distribution of the random variable $Z = \int_0^1 s\, dX_s$?

(b) Show that $Z = B_1 + \dfrac{\mu}{2} - \int_0^1 B_s\, ds$.