Exercise 8 10/11/2025

Exercice 1.

Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. We consider a partition sequence $\left(t_0^{(n)}, \dots, t_n^{(n)}\right)$ in $[0, T]$ such that $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = T$ and $\lim_{n\to\infty} \max_{1\leqslant i\leqslant n} \left(t_i^{(n)} - t_{i-1}^{(n)}\right) = 0$. Show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} g(B_{t_{i-1}^{(n)}}) \left(B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}}\right)^2 = \int_0^T g(B_s)\, ds, \text{ in } L^2.$$

Exercice 2.

Let $(B_t, t \in \mathbb{R}_+)$ be a standard Brownian motion and $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. For all $n \in \mathbb{N}$ and $i = 0, 1, \dots, 2^n$, we define $t_i^{(n)} = it2^{-n}$. Let Λ_i^n be random variables with values in $[0, 1]$. Show that

$$\sum_{i=1}^{2^n} \left[g\left(B_{t_{i-1}^{(n)}} + \Lambda_i^n (B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}})\right) - g\left(B_{t_{i-1}^{(n)}}\right)\right] \left(B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}}\right)^2$$

converge to 0 in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Exercice 3.

Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion.

(a) Using Itô formula to show that $(B_t^2 - t,\, t \in \mathbb{R}_+)$ is a martingale (cf. Exercise 3, Q. 2).

(b) Using Itô formula to show that

$$X_t = B_t^4 - 6\int_0^t B_s^2\, ds$$

is a martingale.